Problem 09709/12/O/N/23/ Q1

The coefficient of x^3 in the expansion of $(3 + 2ax)^5$ is six times the coefficient of x^2 in the expansion of $(2 + ax)^6$.

Find the value of the constant a. [4]

Sol

$^5C_r \, 3^{5-r} \, (2ax)^r$

$x^r = x^3$

$r = 3$

$^5C_3 \, 3^2 \, 2^3 a^3 x^3$

$10 \times 9 \times 8 \times a^3$

$720a^3$

$^6C_r \, 2^{6-r} \, (ax)^r$

$x^r = x^2$

$r = 2$

$^6C_2 \, 2^4 \, a^2 x^2$

$15 \times 16 \, a^2$

$240a^2$

$720 a^{\cancel{3}} = 6 \times 240 \cancel{a^2}$

$a = \dfrac{6 \times 240}{720} = 2$

Problem 09709/12/O/N/23/ Q2

Find the exact solution of the equation

$$\tfrac{1}{6}\pi + \tan^{-1}(4x) = -\cos^{-1}\left(\tfrac{1}{2}\sqrt{3}\right). \qquad [2]$$

Sol

$$\tan^{-1}(4x) = -\cos^{-1}\left(\tfrac{1}{2}\sqrt{3}\right) - \frac{\pi}{6}$$

$$= -\frac{1}{3}\pi$$

$$4x = \tan\left(-\tfrac{1}{3}\pi\right) = -\sqrt{3}$$

$$x = -\frac{\sqrt{3}}{4}$$

Problem 09709/12/O/N/23/ Q3

The equation of a curve is such that $\frac{dy}{dx} = \frac{1}{2}x + \frac{72}{x^4}$. The curve passes through the point $P(2, 8)$.

(a) Find the equation of the normal to the curve at P. [2]

(b) Find the equation of the curve. [4]

<u>Sol</u> (a)

$$\left(\frac{dy}{dx}\right)_{x=2} = \frac{1}{2}\times 2 + \frac{72}{(2)^4} = 1 + \frac{72}{16}$$

$$= \frac{88}{16} = \frac{11}{2}$$

Gradient of normal $= -\frac{2}{11}$

$$y - 8 = -\frac{2}{11}(x-2)$$

$$= -\frac{2}{11}x + \frac{4}{11} + 8$$

$$y = -\frac{2}{11}x + \frac{92}{11}$$

(b)

$$\frac{dy}{dx} = \frac{1}{2}x + \frac{72}{x^4} = \frac{1}{2}x + 72x^{-4}$$

$$\int dy = \int (\tfrac{1}{2}x + 72x^{-4})\,dx$$

$$y = \frac{1}{2}\frac{x^2}{2} - \frac{72}{3}x^{-3} + c$$

$P(2,8)$

$$8 = \frac{1}{\cancel{2}}\frac{2^{\cancel{2}}}{\cancel{2}} - \frac{72}{3}(2)^{-3} + c$$

$$8 = 1 - \frac{\cancel{72}\,\cancel{9}\,3}{\cancel{3}\times\cancel{8}} + c$$

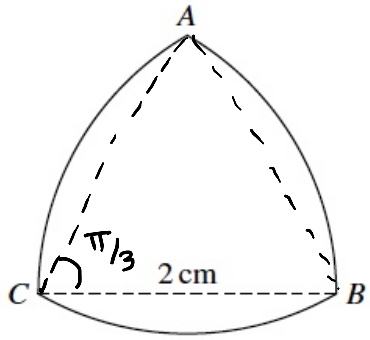
$$8 = 1 - 3 + c$$

$$c = 8 + 3 - 1 = 10$$

$$y = \frac{1}{4}x^2 - \frac{72}{3x^3} + 10$$

$$y = \frac{1}{4}x^2 - \frac{24}{x^3} + 10$$

Problem 09709/12/O/N/23/ Q4



The diagram shows the shape of a coin. The three arcs AB, BC and CA are parts of circles with centres C, A and B respectively. ABC is an equilateral triangle with sides of length 2 cm.

(a) Find the perimeter of the coin. [2]

(b) Find the area of the face ABC of the coin, giving the answer in terms of π and $\sqrt{3}$. [4]

Sol (a) Arc length $= 3 \times 2 \times \frac{\pi}{3} = 2\pi$

(b) Area of the Coin $=$ 3 × Area of sector − Area of equilateral △

$= \cancel{3} \times \frac{1}{\cancel{2}} \times \cancel{2}^{2}\ \frac{\pi}{\cancel{3}} - \frac{\sqrt{3}}{\cancel{4}} 4$

$= 2\pi - \sqrt{3}$

Problem 09709/12/O/N/23/ Q5

The first, second and third terms of a geometric progression are $\sin\theta$, $\cos\theta$ and $2 - \sin\theta$ respectively, where θ radians is an acute angle.

(a) Find the value of θ. [3]

(b) Using this value of θ, find the sum of the first 10 terms of the progression. Give the answer in the form $\frac{b}{\sqrt{c}-1}$, where b and c are integers to be found. [3]

Sol (a)

$$\frac{\cos\theta}{\sin\theta} = \frac{2-\sin\theta}{\cos\theta}$$

$$\cos^2\theta = 2\sin\theta - \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 2\sin\theta$$

$$1 = 2\sin\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

(b)

$$a = \sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$r = \frac{\cos\theta}{\sin\theta} = \frac{\cos \pi/6}{\sin \pi/6} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$S_{10} = \frac{a(r^n-1)}{r-1}$$

$$= \frac{1}{2} \times \frac{(\sqrt{3})^{10}-1}{\sqrt{3}-1} = \frac{1}{\cancel{2}} \times \frac{\overset{121}{\cancel{242}}}{\sqrt{3}-1}$$

$$= \frac{121}{\sqrt{3}-1}$$

Problem 09709/12/O/N/23/ Q6

The equation of a curve is $y = x^2 - 8x + 5$.

(a) Find the coordinates of the minimum point of the curve. [2]

The curve is stretched by a factor of 2 parallel to the y-axis and then translated by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(b) Find the coordinates of the minimum point of the transformed curve. [2]

(c) Find the equation of the transformed curve. Give the answer in the form $y = ax^2 + bx + c$, where a, b and c are integers to be found. [4]

Sol (a)

$$y = x^2 - 8x + 5$$
$$= x^2 - 8x + 16 - 16 + 5$$
$$= (x-4)^2 - 11$$

$(4, -11)$

(b)

$$y = 2(x - 4 - 4)^2 - 11 \times 2 + 1$$
$$= 2(x-8)^2 - 21$$

$(8, -21)$

(c)

$$y = 2(x-8)^2 - 21$$
$$= 2(x^2 - 16x + 64) - 21$$
$$= 2x^2 - 32x + 128 - 21$$
$$= 2x^2 - 32x + 107$$

Problem 09709/12/O/N/23/ Q7

(a) Verify the identity $(2x - 1)(4x^2 + 2x - 1) \equiv 8x^3 - 4x + 1$. [1]

(b) Prove the identity $\dfrac{\tan^2\theta + 1}{\tan^2\theta - 1} \equiv \dfrac{1}{1 - 2\cos^2\theta}$. [3]

(c) Using the results of **(a)** and **(b)**, solve the equation

$$\frac{\tan^2\theta + 1}{\tan^2\theta - 1} = 4\cos\theta,$$

for $0° \leqslant \theta \leqslant 180°$. [5]

Sol (a) $2x(4x^2+2x-1) - 1(4x^2+2x-1)$

$8x^3 + \cancel{4x^2} - 2x - \cancel{4x^2} - 2x + 1$

$8x^3 - 4x + 1$

(b) $\dfrac{\tan^2\theta+1}{\tan^2\theta-1} = \dfrac{\frac{\sin^2\theta}{\cos^2\theta}+1}{\frac{\sin^2\theta}{\cos^2\theta}-1}$

$$\frac{\frac{\sin^2\theta+\cos^2\theta}{\cancel{\cos^2\theta}}}{\frac{\sin^2\theta-\cos^2\theta}{\cancel{\cos^2\theta}}}$$

$$\frac{1}{1-\cos^2\theta-\cos^2\theta}$$

$= \dfrac{1}{1-2\cos^2\theta}$ RHS hence proved.

(c) $\dfrac{1}{1-2\cos^2\theta} = 4\cos\theta$

$1 = 4\cos\theta - 8\cos^3\theta$

$8\cos^3\theta - 4\cos\theta + 1 = 0$

$$(2\cos\theta - 1)(4\cos^2\theta + 2\cos\theta - 1) = 0$$

$$2\cos\theta - 1 = 0$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\frac{1}{2}$$

$$= 60^\circ$$

$$4\cos^2\theta + 2\cos\theta - 1 = 0$$

let $\cos\theta = t$

$$4t^2 + 2t - 1 = 0$$

$$t = 0.309 \qquad t = -0.809$$

$$\cos\theta = 0.309$$

$$\theta = \cos^{-1} 0.309$$

$$= 72^\circ$$

$$\cos\theta = 0.809$$

$$\theta = \cos^{-1} 0.809$$

$$= 36.00^\circ$$

$$\theta = 180 - 36.00$$

$$= 143.9^\circ$$

$$= 144^\circ$$

Problem 09709/12/O/N/23/ Q8

Functions f and g are defined by

$$f(x) = (x + a)^2 - a \text{ for } x \leqslant -a,$$
$$g(x) = 2x - 1 \text{ for } x \in \mathbb{R},$$

where a is a positive constant.

(a) Find an expression for $f^{-1}(x)$. [3]

(b) (i) State the domain of the function f^{-1}. [1]

(ii) State the range of the function f^{-1}. [1]

(c) Given that $a = \frac{7}{2}$, solve the equation $gf(x) = 0$. [3]

Sol (a) $f(x) = (x+a)^2 - a$

let $y = (x+a)^2 - a$

$y + a = (x+a)^2$

$x + a = \pm\sqrt{y+a}$

$x = -a \pm \sqrt{y+a}$

$f^{-1}(x) = -a \pm \sqrt{x+a}$

(b) (i) $x + a \geqslant 0$

$x \geqslant -a$

(ii) $f^{-1}(x) \leqslant -a$

(c) $gf(x) = 0$

$g((x+a)^2 - a) = 0$

$2((x+a)^2 - a) - 1 = 0$

$2(x+a)^2 - 2a - 1 = 0$

$a = \frac{7}{2}$ $\quad 2\left(x^2 + 2 \times \frac{7}{2} \times x + \frac{49}{4}\right) - 2 \times \frac{7}{2} - 1 = 0$

$2\left(x^2 + 7x + \frac{49}{4}\right) - 7 - 1 = 0$

$$2x^2 + 14x + \frac{49}{2} - 8 = 0$$

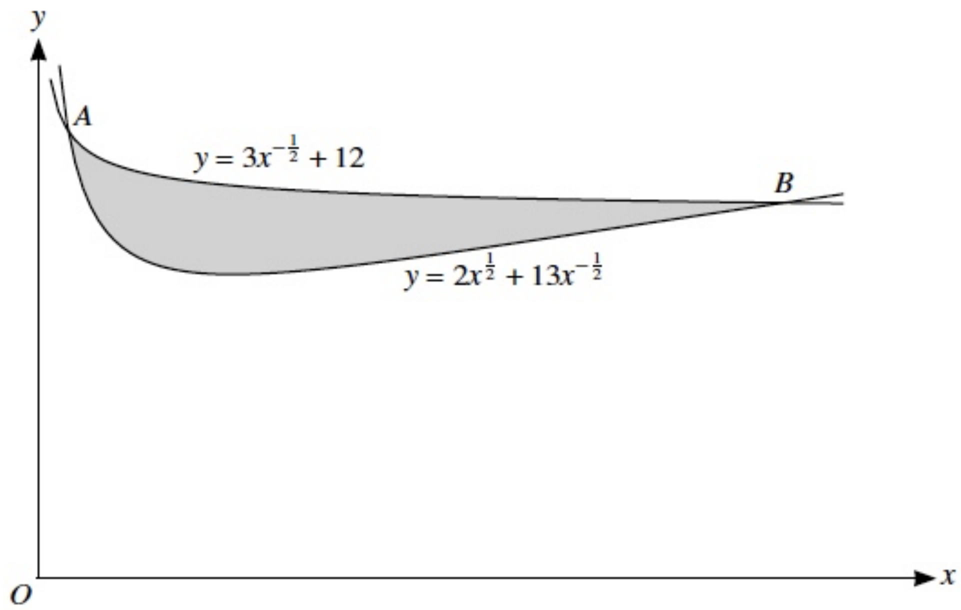
$$4x^2 + 28x + 49 - 16 = 0$$

$$4x^2 + 28x + 33 = 0$$

$$x = -\frac{3}{2} \quad x = -\frac{11}{2}$$

$$x = -\frac{11}{2}$$

Problem 09709/12/O/N/23/ Q9



The diagram shows curves with equations $y = 2x^{\frac{1}{2}} + 13x^{-\frac{1}{2}}$ and $y = 3x^{-\frac{1}{2}} + 12$. The curves intersect at points A and B.

(a) Find the coordinates of A and B. [4]

(b) Hence find the area of the shaded region. [5]

Sol (a) $2x^{\frac{1}{2}} + 13x^{-\frac{1}{2}} = 3x^{-\frac{1}{2}} + 12$

$$2x^{\frac{1}{2}} = -10x^{-1/2} + 12$$

$$2x^{1/2} = -\frac{10}{x^{1/2}} + 12$$

let $x^{1/2} = t$

$$2t = -\frac{10}{t} + 12$$

$$2t^2 = -10 + 12t$$

$$2t^2 - 12t + 10 = 0$$

$$t^2 - 6t + 5 = 0$$

$$t = 5 \qquad t = 1$$

$$x = 25 \qquad x = 1$$

$$y = 3x^{-1/2} + 12$$

$x = 25 \quad y = 3 \times 25^{-1/2} + 12$

$= \frac{3}{5} + 12 = \frac{63}{5}$

$\left(25, \frac{63}{5}\right)$

$x = 1 \quad y = 3 \times (1)^{-1/2} + 12$

$= 15$

$(1, 15)$

(b) $x = 1, \ x = 25$

$$\int_1^{25} \left(3x^{-1/2} + 12 - 2x^{1/2} - 13x^{-1/2}\right) dx$$

$$\left[3\frac{x^{1/2}}{1/2} + 12x - 2\frac{x^{3/2}}{3/2} - 13\frac{x^{1/2}}{1/2}\right]_1^{25}$$

$$\left[6(25)^{1/2} + 12(25) - \frac{4}{3}(25)^{3/2} - 26(25)^{1/2}\right]$$

$$- \left[6(1)^{1/2} + 12(1) - \frac{4}{3}(1)^{3/2} - 26(1)^{1/2}\right]$$

$$= \frac{128}{3}$$

Problem 09709/12/O/N/23/ Q10

The equation of a curve is $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = (4x - 3)^{\frac{5}{3}} - \frac{20}{3}x$.

(a) Find the x-coordinates of the stationary points of the curve and determine their nature. [6]

(b) State the set of values for which the function f is increasing. [1]

Sol (a)

$$f'(x) = \frac{5}{3}(4x-3)^{2/3} \times 4 - \frac{20}{3}$$

$f'(x) = 0$ As stationary point

$$\frac{20}{3}(4x-3)^{2/3} - \frac{20}{3} = 0$$

$$\frac{20}{3}\left[(4x-3)^{2/3} - 1\right] = 0$$

$$(4x-3)^{2/3} - 1 = 0$$

$$(4x-3)^{2/3} = 1$$

$$4x - 3 = \pm 1$$

$4x = 4$ $\qquad$ $4x = 2$

$x = 1$ $\qquad$ $x = 1/2$

(b) $\qquad x < \frac{1}{2}$, $x > 1$

Problem 09709/12/O/N/23/ Q11

The coordinates of points A, B and C are $(6, 4)$, $(p, 7)$ and $(14, 18)$ respectively, where p is a constant. The line AB is perpendicular to the line BC.

(a) Given that $p < 10$, find the value of p. [4]

A circle passes through the points A, B and C.

(b) Find the equation of the circle. [3]

(c) Find the equation of the tangent to the circle at C, giving the answer in the form $dx + ey + f = 0$, where d, e and f are integers. [3]

Sol (a) gradient of $AB = \dfrac{7-4}{p-6}$

gradient of $BC = \dfrac{18-7}{14-p}$

$$m_1 = -\frac{1}{m_2}$$

$$\frac{7-4}{p-6} = -\frac{14-p}{18-7}$$

$$3 \times 11 = -(14-p)(p-6)$$

$$33 = -[14p - 84 - p^2 + 6p]$$

$$33 = -[20p - 84 - p^2]$$

$$33 = -20p + 84 + p^2$$

$$p^2 - 20p + 51 = 0$$

$$p = 17 \qquad \underline{\underline{p = 3}}$$

$p < 10$

(b) $\underline{A(6,4)}$ $B(3,7)$ $\underline{C(14,18)}$

Centre $\left(\dfrac{6+14}{2}, \dfrac{4+18}{2}\right)$

$C(10, 11)$

$$\text{radius} = \frac{1}{2}\sqrt{(18-4)^2 + (14-6)^2}$$

$$= \frac{1}{2}\sqrt{196 + 64} \qquad = \sqrt{65}$$

$(x-10)^2 + (y-11)^2 = 65$

(c) gradient between centre $(10,11)$
$c\,(14, 18)$

$$= \frac{18-11}{14-10} = \frac{7}{4}$$

gradient of tangent at c

$$= -\frac{4}{7}$$

$$y - 18 = -\frac{4}{7}(x-14)$$

$$y = -\frac{4}{7}x + \frac{56}{7} + 18$$

$$= -\frac{4}{7}x + 26$$

$$7y + 4x = 182$$

$$4x + 7y - 182 = 0$$