Problem : 09709/13/M/J/24/Q1

Find the coefficient of x^2 in the expansion of

$$(2-5x)(1+3x)^{10}. \qquad [4]$$

Sol $(2-5x)(1+3x)^{10}$

x^2 and x

${}^{10}C_r (1)^{10-r}(3x)^r \quad x^2$

$r = 2$

${}^{10}C_2 (3x)^2 = 45 \times 9$

$= 405x^2$

x^1

$r = 1$

${}^{10}C_1 (3x)^1 = 10 \times 3 = 30x$

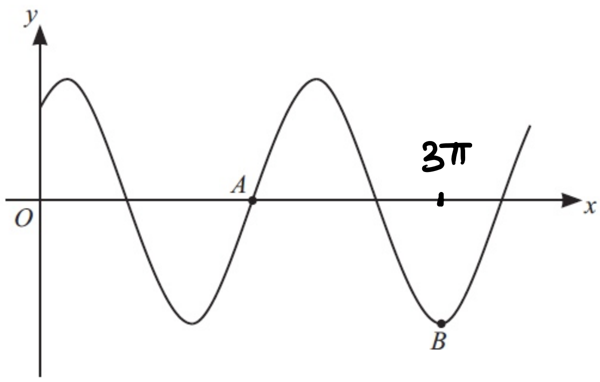
$(2-5x)(30x + 405x^2)$

$810x^2 - 150x^2 = 660x^2$

$= 660$

Problem : 09709/13/M/J/24/Q2

(a)



The diagram shows the curve $y = k\cos(x - \frac{1}{6}\pi)$ where k is a positive constant and x is measured in radians. The curve crosses the x-axis at point A and B is a minimum point.

Find the coordinates of A and B. [3]

(b) Find the exact value of t that satisfies the equation

$$3\sin^{-1}(3t) + 2\cos^{-1}\left(\tfrac{1}{2}\sqrt{2}\right) = \pi. \quad [2]$$

Sol (a)

$$0 = k\cos\left(x - \tfrac{1}{6}\pi\right)$$

$$\cos\left(x - \tfrac{1}{6}\pi\right) = 0$$

$$x - \tfrac{1}{6}\pi = \cos^{-1}(0) = \tfrac{3}{2}\pi$$

$$x = \tfrac{3}{2}\pi + \tfrac{1}{6}\pi = \tfrac{5}{3}\pi$$

$$A\left(\tfrac{5}{3}\pi, 0\right)$$

$$y = -k$$

$$-k = k\cos\left(x - \tfrac{1}{6}\pi\right)$$

$$-1 = \cos\left(x - \tfrac{1}{6}\pi\right)$$

$$x - \tfrac{1}{6}\pi = \cos^{-1}(-1)$$

$$x = 3\pi + \tfrac{1}{6}\pi$$

$$x = \tfrac{19}{6}\pi$$

$$B\left(\tfrac{19}{6}\pi, -k\right)$$

(b)

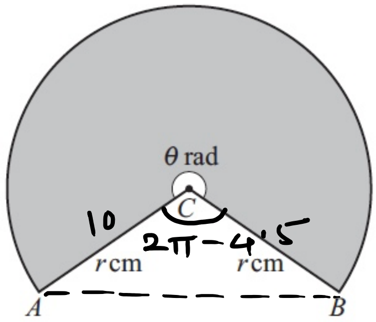
$$3\sin^{-1} 3t = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\sin^{-1} 3t = \frac{\pi}{6}$$

$$3t = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$t = 1/6$$

Problem : 09709/13/M/J/24/Q3



The diagram shows a sector of a circle with centre C. The radii CA and CB each have length r cm and the size of the reflex angle ACB is θ radians. The sector, shaded in the diagram, has a perimeter of 65 cm and an area of 225 cm^2.

(a) Find the values of r and θ. [4]

(b) Find the area of triangle ACB. [2]

Sol (a) $65 = 2r + r\theta$ —(i)

$225 = \frac{1}{2} r^2 \theta$

$450 = r^2 \theta$ —(ii)

$65 = 2r + \cancel{r} \times \frac{450}{r^{\cancel{2}}}$

$65r = 2r^2 + 450$

$2r^2 - 65r + 450 = 0$

$r = 10 \qquad r = 22.5$

$450 = 10^2 \theta$

$\theta = 4.5$

(b) Area of Triangle ACB

$= \frac{1}{2} \times 10 \times 10 \times \sin(2\pi - 4.5)$

$= 48.9\text{ cm}^2$

Problem : 09709/13/M/J/24/Q4

(a) Show that the equation $\cos\theta(7\tan\theta - 5\cos\theta) = 1$ can be written in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are integers to be found. [3]

(b) Hence solve the equation $\cos 2x(7\tan 2x - 5\cos 2x) = 1$ for $0° < x < 180°$. [3]

Sol (a) $\cos\left(7\dfrac{\sin\theta}{\cos\theta} - 5\cos\theta\right) = 1$

$7\sin\theta - 5\cos^2\theta = 1$

$7\sin\theta - 5(1 - \sin^2\theta) = 1$

$7\sin\theta - 5 + 5\sin^2\theta = 1$

$5\sin^2\theta + 7\sin\theta - 6 = 0$

(b) $5\sin^2 2x + 7\sin 2x - 6 = 0$

Let $\sin 2x = t$

$5t^2 + 7t - 6 = 0$

$5t^2 + 10t - 3t - 6 = 0$

$5t(t+2) - 3(t+2) = 0$

$(5t-3)(t+2) = 0$

$t = \dfrac{3}{5}$ $\quad t = -2$

$\sin 2x = \dfrac{3}{5}$ $\quad \sin 2x = -2$ ✗

$2x = \sin^{-1}\dfrac{3}{5}$

$2x = 36.9$ $\quad 2x = 180 - 36.9$

$x = 18.4°$ $\quad x = 71.6°$

Problem : 09709/13/M/J/24/Q5

The equation of a curve is $y = 2x^2 - \frac{1}{2x} + 3$.

(a) Find the coordinates of the stationary point. [3]

(b) Determine the nature of the stationary point. [2]

(c) For positive values of x, determine whether the curve shows a function that is increasing, decreasing or neither. Give a reason for your answer. [2]

Sol (a) $\frac{dy}{dx} = 4x + \frac{1}{2x^2} + 0$

$0 = 4x + \frac{1}{2x^2}$

$0 = 8x^3 + 1$

$8x^3 = -1$

$x^3 = -1/8$

$x = -1/2$

$y = 2\left(-\frac{1}{2}\right)^2 - \frac{1}{\cancel{2} \times -\frac{1}{\cancel{2}}} + 3$

$= \cancel{2} \times \frac{1}{\cancel{4}2} + 1 + 3$

$= 4 + \frac{1}{2} = 9/2$

Coordinate of stationary point $\left(-\frac{1}{2}, \frac{9}{2}\right)$

(b) $\frac{d^2y}{dx^2} = 4 + \frac{1}{2} \times \frac{-2}{x^3}$

$x = -1/2 \quad \frac{d^2y}{dx^2} = 4 - \frac{1}{(-1/2)^3} = 12 > 0$

So nature of stationary point would be minmum.

(c) $\frac{dy}{dx} = 4x + \frac{1}{2x^2}$

$x > 0$ then $\frac{dy}{dx} > 0$

hence function would be increasing.

Problem : 09709/13/M/J/24/Q6

A curve passes through the point $\left(\frac{4}{5}, -3\right)$ and is such that $\frac{dy}{dx} = \frac{-20}{(5x-3)^2}$.

(a) Find the equation of the curve. [4]

(b) The curve is transformed by a stretch in the x-direction with scale factor $\frac{1}{2}$ followed by a translation of $\begin{pmatrix} 2 \\ 10 \end{pmatrix}$.

Find the equation of the new curve. [3]

Sol (a)

$$\frac{dy}{dx} = -20(5x-3)^{-2}$$

$$\int dy = -20\int(5x-3)^{-2}\,dx$$

$$y = -20\,\frac{(5x-3)^{-1}}{-1} \times \frac{1}{5} + c$$

$\left(\frac{4}{5}, -3\right)$ $\quad -3 = +\frac{\cancel{20}^{4}\left(\cancel{5} \times \frac{4}{\cancel{5}} - 3\right)^{-1}}{+\cancel{5}} + c$

$-3 = 4 + c \quad \therefore c = -7$

$$y = 4(5x-3)^{-1} - 7$$

(b)

$$y = \frac{4}{(5x-3)} - 7$$

$x \to 2x \quad x \to (x-2) \quad y \to y+10$

$$y = \frac{4}{[5 \times 2(x-2) - 3]} - 7 + 10$$

$$= \frac{4}{10x - 23} + 3$$

Problem : 09709/13/M/J/24/Q7

The first term of an arithmetic progression is 1.5 and the sum of the first ten terms is 127.5 .

(a) Find the common difference. [2]

(b) Find the sum of all the terms of the arithmetic progression whose values are between 25 and 100. [5]

Sol (a) $a = 1.5$

$S_{10} = 127.5$

$127.5 = \frac{10}{2}\left[2\times 1.5 + (10-1)d\right]$

$127.5 = 5\left[3 + 9d\right]$

$\left[\frac{127.5}{5} - 3\right]\frac{1}{9} = d$

$d = 2.5$

(b) $1.5 + (n-1)2.5 > 25$

$n-1 > \frac{25 - 1.5}{2.5}$

$n > 1 + \frac{47}{5}$

$n > 10.4 \qquad n \approx 10$

$1.5 + (n-1)2.5 < 100$

$n < 1 + \frac{100 - 1.5}{2.5} \qquad n < 40.4$

$n \approx 40$

$S_{40} = \frac{40}{2}\left[2\times 1.5 + (40-1)2.5\right]$

$= 2010$

$S_{10} = 127.5$

$S_{40} - S_{10} = 2010 - 127.5$

$= 1882.5$

Problem : 09709/13/M/J/24/Q8

A circle with equation $x^2 + y^2 - 6x + 2y - 15 = 0$ meets the y-axis at the points A and B. The tangents to the circle at A and B meet at the point P.

Find the coordinates of P. [8]

Sol

$x = 0$ at points A and B

$y^2 + 2y - 15 = 0$

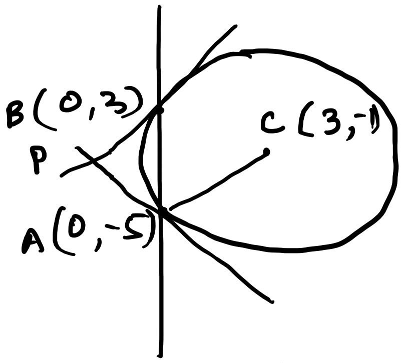
$y^2 + 5y - 3y - 15 = 0$

$y(y+5) - 3(y+5) = 0$

$(y+5)(y-3) = 0$

$y = -5, 3$

$A(0, -5)$ $B(0, 3)$



Centre $(3, -1)$

gradient AC $= \dfrac{-5+1}{0-3} = \dfrac{-4}{-3} = \dfrac{4}{3}$

gradient of tangent at A $= -\dfrac{3}{4}$

Equation of tangent at A

$y + 5 = -\frac{3}{4}(x$

$y = -\frac{3}{4}x - 5$

gradient BC $= \dfrac{3+1}{0-3} = -\dfrac{4}{3}$

gradient of tangent at B $= \dfrac{3}{4}$

Equation of tangent at B

$y - 3 = \frac{3}{4}x$

$y = \frac{3}{4}x + 3$

$-\frac{3}{4}x - 5 = \frac{3}{4}x + 3$

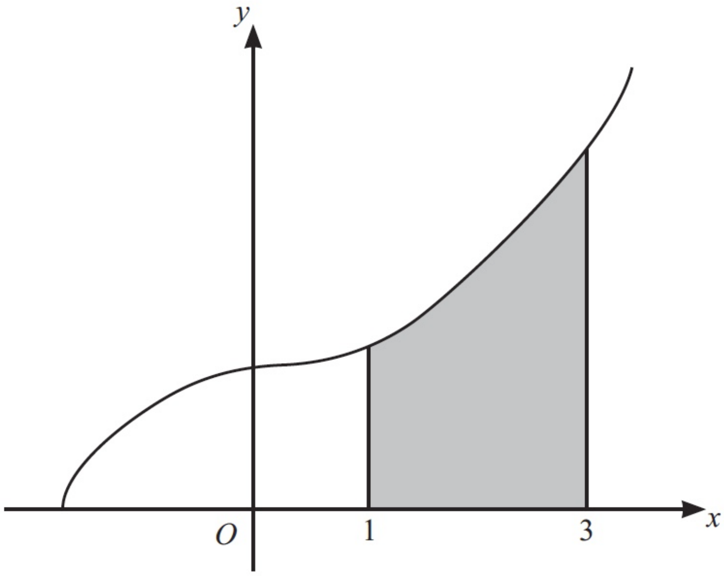
$-5 - 3 = \frac{3}{4}x + \frac{3}{4}x = \frac{6}{4}x$

$-8 = 6/4\, x$ $\quad x = \dfrac{-8 \times 4}{6} = -\dfrac{16}{3}$

$y = \dfrac{\cancel{3}}{\cancel{4}} \times \dfrac{-\cancel{16}}{\cancel{3}} + 3 = -1$

$P\left(-\frac{16}{3}, -1\right)$

Problem : 09709/13/M/J/24/Q9



The diagram shows the curve with equation $y = \sqrt{2x^3 + 10}$.

(a) Find the equation of the tangent to the curve at the point where $x = 3$. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers. [5]

(b) The region shaded in the diagram is enclosed by the curve and the straight lines $x = 1$, $x = 3$ and $y = 0$.

Find the volume of the solid obtained when the shaded region is rotated through 360° about the x-axis. [3]

Sol (a)

$$y = \sqrt{2x^3+10}$$

$$x = 3$$

$$y = \sqrt{2(3)^3+10} = 8$$

$$\frac{dy}{dx} = \frac{1}{2}(2x^3+10)^{-1/2} \times 2 \times 3 \times x^2$$

$$x = 3 \quad \frac{dy}{dx} = \frac{1}{2}(2 \times 3^3 + 10)^{-1/2} \times 6 \times (3)^2$$

$$= \frac{27}{8}$$

So equation of tangent

$$y - 8 = \frac{27}{8}(x-3)$$

$$8y - 64 = 27x - 81$$

$$27x - 8y - 17 = 0$$

(b)

$$V = \pi \int_1^3 (2x^3 + 10)\, dx$$

$$= \pi \left[2\frac{x^4}{4} + 10x \right]_1^3$$

$$= \pi \left[\frac{1}{2} 3^4 + 10 \times 3 - \frac{2}{4} - 10 \right]$$

$$= 60\pi$$

Problem : 09709/13/M/J/24/Q10

The geometric progression $a_1, a_2, a_3, \ldots$ has first term 2 and common ratio r where $r > 0$.
It is given that $\frac{9}{2}a_5 + 7a_3 = 8$.

(a) Find the value of r. [3]

(b) Find the sum of the first 20 terms of the geometric progression. Give your answer correct to 4 significant figures. [2]

(c) Find the sum to infinity of the progression $a_2, a_5, a_8, \ldots$. [3]

Sol (a) $a = 2$

$$\frac{9}{2}(a.r^4) + 7(ar^2) = 8$$

$$\frac{9}{\cancel{2}} \cancel{2}.r^4 + 7.2r^2 = 8$$

$$9r^4 + 14r^2 = 8$$

Let $r^2 = t$

$$9t^2 + 14t - 8 = 0$$

$$9t^2 + 18t - 4t - 8 = 0$$

$$9t(t+2) - 4(t+2) = 0$$

$$(t+2)(9t-4) = 0$$

$t = -2$ $\quad$ $t = \frac{4}{9}$

$r^2 = -2$ ✗ $\quad$ $r^2 = \frac{4}{9}$

$$r = \left(\frac{4}{9}\right)^{1/2}$$

$$r = \frac{2}{3}$$

(b) $$S_{20} = \frac{2\left(1 - \left(\frac{2}{3}\right)^{20}\right)}{1 - \frac{2}{3}}$$

$$= 6\left(1 - \left(\frac{2}{3}\right)^{20}\right)$$

$$= 5.998$$

(c) $a_2 = ar$

$= 2 \times \frac{2}{3} = \frac{4}{3}$

$a_5 = ar^4$

$= 2 \times \left(\frac{2}{3}\right)^4 = \frac{32}{81}$

$r = \frac{32}{81} \times \frac{3}{4} = \frac{8}{27}$

$S_\infty = \frac{a}{1-r} = \frac{4/3}{1-\frac{8}{27}} = \frac{36}{19}$

Problem : 09709/13/M/J/24/Q11

The function f is defined by $f(x) = 10 + 6x - x^2$ for $x \in \mathbb{R}$.

(a) By completing the square, find the range of f. [3]

The function g is defined by $g(x) = 4x + k$ for $x \in \mathbb{R}$ where k is a constant.

(b) It is given that the graph of $y = g^{-1}f(x)$ meets the graph of $y = g(x)$ at a single point P.

Determine the coordinates of P. [6]

Sol (a) $f(x) = 10 + 6x - x^2$

$= -x^2 + 6x + 10$

$= -(x^2 - 6x) + 10$

$= -(\underline{x^2 - 6x + 9} - 9) + 10$

$= -(x-3)^2 + 9 + 10$

$= -(x-3)^2 + 19$

$f(x) \leq 19$

(b) $g(x) = 4x + K$ ✓

let $y = 4x + K$

$x = \frac{1}{4}(y - K)$

$g^{-1}(x) = \frac{1}{4}(x - K)$

$g^{-1}f(x) = g(x)$

$\frac{1}{4}(-x^2 + 6x + 10 - K) = 4x + K$

$-x^2 + 6x + 10 - K = 16x + 4K$

✓ $x^2 + 10x + 5K - 10 = 0$

$b^2 - 4ac = 0 \quad 10^2 - 4(5K - 10) = 0$

$100 = 4(5K - 10) \quad \therefore K = 7$

$x^2 + 10x + 25 = 0 \quad (x+5)^2 = 0$

$x = -5$

$y = 4 \times -5 + 7 = -13$

$P(-5, -13)$