Problem 09709/13/O/N/23/ Q1

A curve is such that its gradient at a point (x, y) is given by $\frac{dy}{dx} = x - 3x^{-\frac{1}{2}}$. It is given that the curve passes through the point $(4, 1)$.

Find the equation of the curve. [4]

Sol

$$\int dy = \int x - 3x^{-\frac{1}{2}} \, dx$$

$$y = \frac{x^2}{2} - 3\,\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$y = \frac{x^2}{2} - 6x^{1/2} + c$$

$$1 = \frac{4^2}{2} - 6(4)^{1/2} + c$$

$$1 = 8 - 12 + c$$

$$c = 1 + 12 - 8 = 5$$

$$y = \frac{x^2}{2} - 6x^{1/2} + 5$$

Problem 09709/13/O/N/23/ Q2

The circle with equation $(x-3)^2 + (y-5)^2 = 40$ intersects the y-axis at points A and B.

(a) Find the y-coordinates of A and B, expressing your answers in terms of surds. [2]

(b) Find the equation of the circle which has AB as its diameter. [2]

Sol (a) $x = 0$

$$(0-3)^2 + (y-5)^2 = 40$$

$$9 + (y-5)^2 = 40$$

$$(y-5)^2 = 40 - 9 = 31$$

$$y = 5 \pm \sqrt{31}$$

$A(0, 5+\sqrt{31})$ $B(0, 5-\sqrt{31})$

(b)

$B(0, 5-\sqrt{31}$ —— $A(0, 5+\sqrt{31})$

$$\text{Radius} = \sqrt{(0-0)^2 + (5-\sqrt{31} - 5 - \sqrt{31})^2}$$

$$= \sqrt{0 + (2\sqrt{31})^2}$$

$$= \frac{2\sqrt{31}}{2} = \sqrt{31}$$

$$\text{Centre} = \left(\frac{0+0}{2}, \frac{5+\cancel{\sqrt{31}}+5-\cancel{\sqrt{31}}}{2}\right)$$

$$= (0, 5)$$

Eq. of Circle $x^2 + (y-5)^2 = 31$

Problem 09709/13/O/N/23/ Q3

(a) Show that the equation

$$5\cos\theta - \sin\theta\tan\theta + 1 = 0$$

may be expressed in the form $a\cos^2\theta + b\cos\theta + c = 0$, where a, b and c are constants to be found. [3]

(b) Hence solve the equation $5\cos\theta - \sin\theta\tan\theta + 1 = 0$ for $0 < \theta < 2\pi$. [4]

Sol (a) $5\cos\theta - \sin\theta \times \dfrac{\sin\theta}{\cos\theta} + 1 = 0$

$5\cos^2\theta - \sin^2\theta + \cos\theta = 0$

$5\cos^2\theta - 1 + \cos^2\theta + \cos\theta = 0$

$6\cos^2\theta + \cos\theta - 1 = 0$

(b) $6\cos^2\theta + \cos\theta - 1 = 0$

let $\cos\theta = t$

$6t^2 + t - 1 = 0$

$t = \dfrac{1}{3}$ $\quad t = -\dfrac{1}{2}$

$\cos\theta = \dfrac{1}{3}$ $\quad \cos\theta = -\dfrac{1}{2}$

$\theta = \cos^{-1}\dfrac{1}{3}$ $\quad \theta = \cos^{-1}\dfrac{1}{2}$

$\theta = 1.23$ $\quad = \dfrac{\pi}{3}$

$\theta = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

$\theta = \pi + \dfrac{\pi}{3}$

$= \dfrac{4\pi}{3}$

Problem 09709/13/O/N/23/ Q4

(a) Expand the following in ascending powers of x up to and including the term in x^2.

(i) $(1+2x)^5$. [1]

(ii) $(1-ax)^6$, where a is a constant. [2]

In the expansion of $(1+2x)^5(1-ax)^6$, the coefficient of x^2 is -5.

(b) Find the possible values of a. [4]

Sol (a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$

(i) $(1+2x)^5 = 1 + 10x + \frac{5\times 4^2}{\cancel{2}}\times(2x)^2$

$= 1 + 10x + 40x^2$

(ii) $(1-ax)^6 = 1 + 6(-ax) + \frac{\overset{3}{\cancel{6}}\times 5}{\cancel{2}}(-ax)^2$

$= 1 - 6ax + 15a^2x^2$

(b) $(1+2x)^5(1-ax)^6 = (1 + 10x + 40x^2) \times (1 - 6ax + 15a^2x^2)$

$= 40x^2 - 60ax^2 + 15a^2x^2$

$= (40 - 60a + 15a^2)x^2$

$-5 = 40 - 60a + 15a^2$

$15a^2 - 60a + 45 = 0$

$a^2 - 4a + 3 = 0$

$a = 3 \quad a = 1$

Problem 09709/13/O/N/23/ Q5

The first, second and third terms of a geometric progression are $2p + 6$, $5p$ and $8p + 2$ respectively.

(a) Find the possible values of the constant p. [3]

(b) One of the values of p found in **(a)** is a negative fraction.

Use this value of p to find the sum to infinity of this progression. [4]

Sol (a)

$$\frac{8p+2}{5p} = \frac{5p}{2p+6}$$

$$(5p)^2 = (8p+2)(2p+6)$$

$$25p^2 = 16p^2 + 48p + 4p + 12$$

$$9p^2 - 52p - 12 = 0$$

$$p = 6 \quad p = -\frac{2}{9}$$

(b)

$$p = -\frac{2}{9} \qquad S_\infty = \frac{a}{1-r}$$

$$a = 2 \times -\frac{2}{9} + 6 = \frac{50}{9}$$

$$r = \frac{5 \times -\frac{2}{9}}{\frac{50}{9}} = \frac{-10}{50} = -\frac{1}{5}$$

$$S_\infty = \frac{\frac{50}{9}}{1-(-\frac{1}{5})}$$

$$= \frac{\frac{50}{9}}{\frac{6}{5}} = \frac{50}{9} \times \frac{5}{6} = \frac{125}{27}$$

Problem 09709/13/O/N/23/ Q6

A line has equation $y = 6x - c$ and a curve has equation $y = cx^2 + 2x - 3$, where c is a constant. The line is a tangent to the curve at point P.

Find the possible values of c and the corresponding coordinates of P. [7]

Sol

$cx^2 + 2x - 3 = 6x - c$

$cx^2 + 2x - 6x - 3 + c = 0$

✓ $cx^2 - 4x + c - 3 = 0$

$b^2 - 4ac = 0$ As line is tangent to the Curve

$(-4)^2 - 4 \times c \times (c-3) = 0$

$16 - 4c(c-3) = 0$

$16 - 4c^2 + 12c = 0$

$4c^2 - 12c - 16 = 0$

$c^2 - 3c - 4 = 0$

$c = 4 \quad c = -1$

$c = 4$

$y = 6x - 4 \quad y = 4x^2 + 2x - 3$

$4x^2 - 4x + 1 = 0$

$(2x-1)^2 = 0 \quad x = 1/2 \quad y = \overset{3}{\cancel{6}} \times \frac{1}{\cancel{2}} - 4 = -1$

$\left(\frac{1}{2}, -1\right)$

$c = -1 \quad -x^2 - 4x - 4 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0 \quad x = -2 \quad y = 6 \times -2 + 1$

$= -11$

$(-2, -11)$

Problem 09709/13/O/N/23/ Q7

The function f is defined by $f(x) = 1 + \dfrac{3}{x-2}$ for $x > 2$.

(a) State the range of f. [1]

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = 2x - 2$ for $x > 0$.

(c) Obtain a simplified expression for $gf(x)$. [2]

Sol (a) $f(x) > 1$

(b) let $y = 1 + \dfrac{3}{x-2}$

$$y - 1 = \frac{3}{x-2}$$

$$x - 2 = \frac{3}{y-1}$$

$$x = 2 + \frac{3}{y-1}$$

$$f^{-1}(x) = 2 + \frac{3}{x-1}$$

$x > 1$

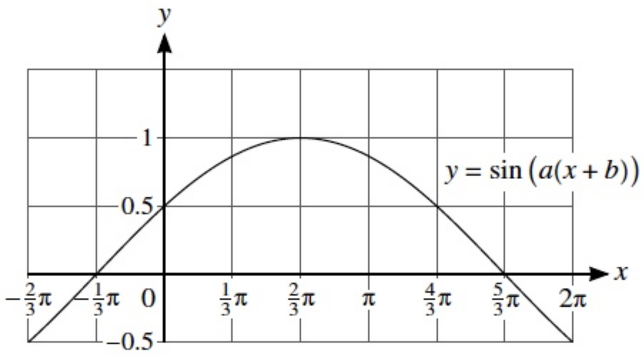
(c) $g(x) = 2x - 2 \quad f(x) = 1 + \dfrac{3}{x-2}$

$$g(f(x)) = 2\left(1 + \frac{3}{(x-2)}\right) - 2$$

$$= \cancel{2} + \frac{6}{x-2} - \cancel{2}$$

$$= \frac{6}{x-2}$$

Problem 09709/13/O/N/23/ Q8



The diagram shows part of the graph of $y = \sin(a(x + b))$, where a and b are positive constants.

(a) State the value of a and one possible value of b. [2]

Another curve, with equation $y = f(x)$, has a single stationary point at the point (p, q), where p and q are constants. This curve is transformed to a curve with equation

$$y = -3f\left(\tfrac{1}{4}(x + 8)\right).$$

(b) For the transformed curve, find the coordinates of the stationary point, giving your answer in terms of p and q. [3]

Sol (a) $\frac{5}{3}\pi - \left(-\frac{\pi}{3}\right) = \frac{6\pi}{3} = 2\pi$

period of graph $= 4\pi$

$a = \frac{2\pi}{\text{period}} = \frac{2\pi}{4\pi} = \frac{1}{2}$

$b = \frac{1}{3}\pi$

(b) x - coordinate $= 4p$

y - coordinate $= -3q$

Problem 09709/13/O/N/23/ Q9

A curve has equation $y = 2x^{\frac{1}{2}} - 1$.

(a) Find the equation of the normal to the curve at the point $A\ (4,\ 3)$, giving your answer in the form $y = mx + c$. [3]

A point is moving along the curve $y = 2x^{\frac{1}{2}} - 1$ in such a way that at A the rate of increase of the x-coordinate is $3\,\text{cm s}^{-1}$.

(b) Find the rate of increase of the y-coordinate at A. [2]

At A the moving point suddenly changes direction and speed, and moves down the normal in such a way that the rate of decrease of the y-coordinate is constant at $5\,\text{cm s}^{-1}$.

(c) As the point moves down the normal, find the rate of change of its x-coordinate. [3]

Sol (a) $\frac{dy}{dx} = \cancel{2} \times \frac{1}{\cancel{2}} \times x^{\frac{1}{2}-1} - 0 = x^{-1/2}$

$\frac{dy}{dx} = (4)^{-1/2} = \frac{1}{2}$

gradient of Normal $= -2$

$y - 3 = -2(x - 4)$

$y - 3 = -2x + 8$

$y = -2x + 11$

(b) $\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = ?$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{1}{2} \times 3$

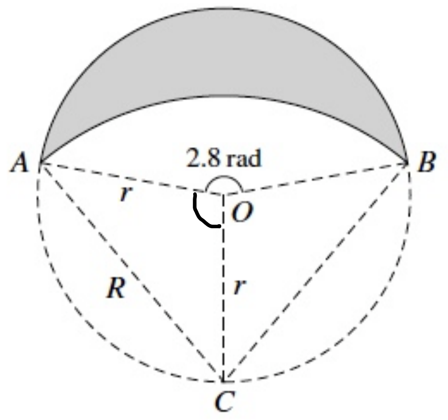
$\frac{dy}{dt} = \frac{3}{2}$

(c) $\frac{dy}{dt} = \left(\frac{dy}{dx}\right)_{\text{normal}} \times \frac{dx}{dt}$

$-5 = -2 \times \frac{dx}{dt}$

$\frac{dx}{dt} = 5/2$

Problem 09709/13/O/N/23/ Q10



The diagram shows points A, B and C lying on a circle with centre O and radius r. Angle AOB is 2.8 radians. The shaded region is bounded by two arcs. The upper arc is part of the circle with centre O and radius r. The lower arc is part of a circle with centre C and radius R.

(a) State the size of angle ACO in radians. [1]

(b) Find R in terms of r. [1]

(c) Find the area of the shaded region in terms of r. [7]

Sol (a) $\angle ACO = \frac{\angle AOB}{4} = \frac{2.8}{4} = 0.7$ radian

(b) $\triangle AOC$

$\angle AOC = \frac{2\pi - 2.8}{2} = 1.741$

$R = \sqrt{r^2 + r^2 - 2r^2 \cos 1.741}$

$= 1.53r$

(c) Area of sector CABC

$= \frac{1}{2} R^2 \, 1.4 = \frac{1}{2} (1.53)^2 r^2 \, 1.4$

$= 1.638 r^2$

Area of $\triangle$ COA

$= \frac{1}{2} r^2 \sin\left(\frac{2\pi - 2.8}{2}\right)$

$= \frac{1}{2} r^2 \times 0.985$

Area of CAOBC $= 0.985 r^2$

Area of region AOBA $= 1.638 r^2 - 0.985 r^2$

$= 0.653$

Area of sector AOBA

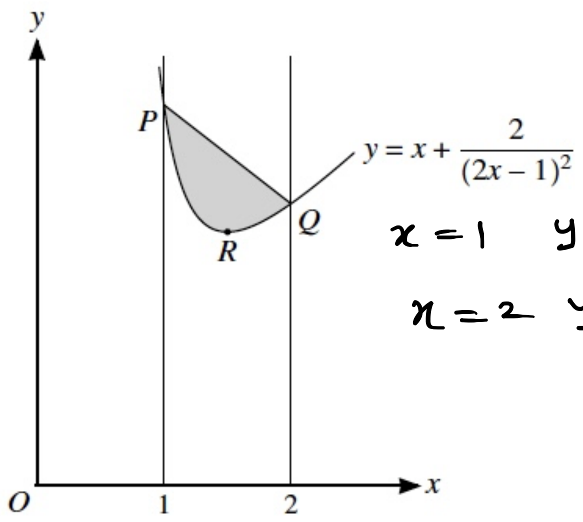
$= \frac{1}{2} r^2 \, 2.8 = 1.4 r^2$

Area of shaded region

$= 1.4r^2 - 0.653r^2$

$= 0.747 r^2$

Problem 09709/13/O/N/23/ Q11



$x = 1 \quad y = 1 + \dfrac{2}{(2\times1-1)^2} = 1 + 2 = 3$

$x = 2 \quad y = 2 + \dfrac{2}{(2\times2-1)^2} = 2 + \dfrac{2}{9}$

$= \dfrac{20}{9}$

The diagram shows part of the curve with equation $y = x + \dfrac{2}{(2x-1)^2}$. The lines $x = 1$ and $x = 2$ intersect the curve at P and Q respectively and R is the stationary point on the curve.

(a) Verify that the x-coordinate of R is $\frac{3}{2}$ and find the y-coordinate of R. [4]

(b) Find the exact value of the area of the shaded region. [6]

Sol (a)

$$y = x + \frac{2}{(2x-1)^2}$$

$$\frac{dy}{dx} = 1 + 2 \times \frac{-2}{(2x-1)^3} \times 2$$

since R is stationary point hence $\frac{dy}{dx} = 0$

$$0 = 1 - \frac{8}{(2x-1)^3}$$

$$(2x-1)^3 = 8$$

$$2x - 1 = 2$$

$$2x = 3$$

$$\therefore \; x = 3/2$$

$$y = \frac{3}{2} + \frac{2}{(\cancel{2}\times\frac{3}{\cancel{2}}-1)^2} = \frac{3}{2} + \frac{2}{4} = \frac{3}{2} + \frac{1}{2} = 2$$

$R\left(\frac{3}{2}, 2\right)$

(b) Eq of line PQ $\quad P(1,3) \quad Q(2, \frac{20}{9})$

$$\text{gradient} = \frac{\frac{20}{9} - 3}{2-1} = -\frac{7}{9}$$

$$y - 3 = -\frac{7}{9}(x-1) = -\frac{7}{9}x + \frac{7}{9}$$

$$y = -\frac{7}{9}x + \frac{7}{9} + 3 = -\frac{7}{9}x + \frac{34}{9}$$

Area of shaded region

$$= \int_1^2 \left[-\frac{7}{9}x + \frac{34}{9} - x - \frac{2}{(2x-1)^2} \right] dx$$

$$= \left[-\frac{7}{9} \times \frac{x^2}{2} + \frac{34}{9}x - \frac{x^2}{2} + \frac{2}{2x-1} \times \frac{1}{2} \right]_1^2$$

$$= \left[-\frac{7}{18}(2)^2 + \frac{34}{9}(2) - \frac{(2)^2}{2} + \frac{2}{2\times 2-1} \times \frac{1}{2} \right]$$
$$- \left[-\frac{7}{18}(1)^2 + \frac{34}{9}(1) - \frac{(1)^2}{2} + \frac{2}{2\times 1-1} \times \frac{1}{2} \right]$$

$$= \frac{4}{9}$$