Problem : 09709/31/O/N/23/Q1

Find the exact coordinates of the points on the curve $y = \dfrac{x^2}{1-3x}$ at which the gradient of the tangent is equal to 8. [5]

Sol By using quotient rule

$$\frac{dy}{dx} = \frac{(1-3x)(2x) - x^2(-3)}{(1-3x)^2}$$

$$8(1-3x)^2 = 2x - 6x^2 + 3x^2$$

$$8(1-6x+9x^2) = 2x - 3x^2$$

$$8 - 48x + 72x^2 = 2x - 3x^2$$

$$75x^2 - 50x + 8 = 0$$

$$x = \frac{2}{5} \qquad\qquad x = \frac{4}{15}$$

$$y = \frac{\left(\frac{2}{5}\right)^2}{1-3\left(\frac{2}{5}\right)} \qquad\qquad y = \frac{(4/15)^2}{1-3(4/15)}$$

$$y = -\frac{4}{5} \qquad\qquad y = \frac{16}{45}$$

$$\left(\frac{2}{5}, -\frac{4}{5}\right) \qquad\qquad \left(\frac{4}{5}, \frac{16}{45}\right)$$

Problem : 09709/31/O/N/23/Q2

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2i| \leqslant |z + 2 - i|$ and $0 \leqslant \arg(z + 1) \leqslant \frac{1}{4}\pi$. [4]

Sol

$$x^2 + (y-2)^2 \leq (x+2)^2 + (y-1)^2$$

$$\cancel{x^2} + \cancel{y^2} - 4y + \cancel{4} \leq \cancel{x^2} + 4x + \cancel{4} + \cancel{y^2} + 1 - 2y$$

$$-4y \leq -2y + 4x + 1$$

$$0 \leq 4y - 2y + 4x + 1$$

$$0 \leq 2y + 4x + 1$$

$$2y \geq -4x - 1$$

$$y \geq -2x - \frac{1}{2}$$

$$\text{Arg}(z+1) \geq 0$$

$$\tan^{-1} \frac{y}{x+1} \geq 0$$

$$y \geq (x+1)\tan 0$$

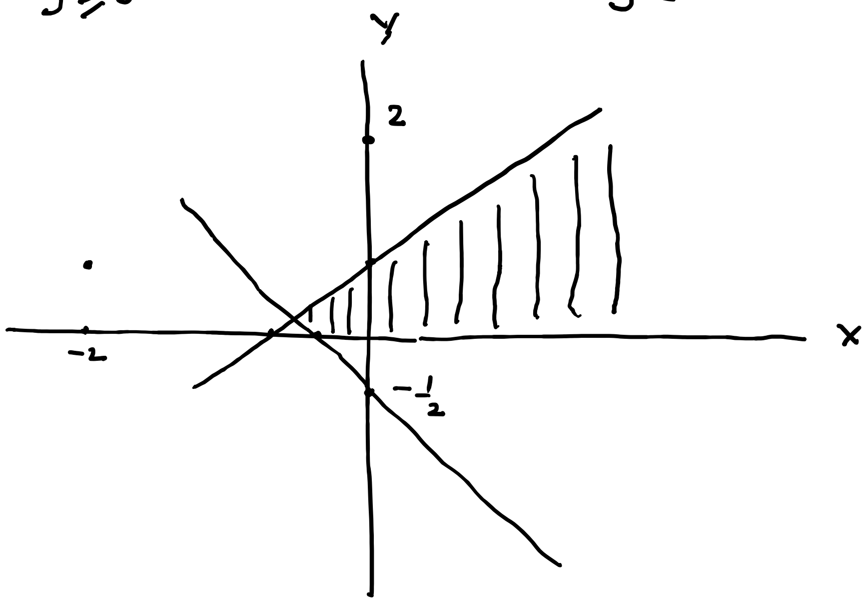
$$y \geq 0$$

$$\text{Arg}(z+1) \leq \frac{\pi}{4}$$

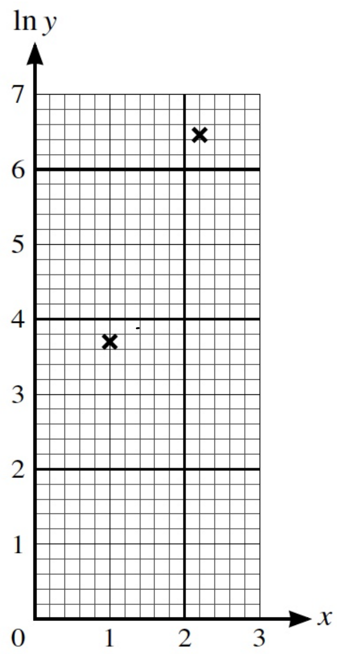
$$\tan^{-1} \frac{y}{x+1} \leq \pi/4$$

$$\frac{y}{x+1} \leq 1$$

$$y \leq x + 1$$



Problem : 09709/31/O/N/23/Q3



The variables x and y are related by the equation $y = ab^x$, where a and b are constants. The diagram shows the result of plotting $\ln y$ against x for two pairs of values of x and y. The coordinates of these points are (1, 3.7) and (2.2, 6.46).

Use this information to find the values of a and b. [4]

Sol

$$y = ab^x$$

$$\ln y = \ln(a.b^x)$$

$$\ln y = \ln a + x \ln b$$

$$3.7 = \ln a + \ln b \quad \text{— (i)}$$

$$6.46 = \ln a + 2.2 \ln b \quad \text{— (ii)}$$

by solving eq (i) and (ii)

$$\ln a = 1.4 \qquad \ln b = 2.3$$

$$a = e^{1.4} \qquad b = e^{2.3}$$

$$a = 4.06 \qquad b = 9.97$$

Problem : 09709/31/O/N/23/Q4

The complex number u is defined by $u = \dfrac{3+2i}{a-5i}$, where a is real.

(a) Express u in the Cartesian form $x + iy$, where x and y are in terms of a. [3]

(b) Given that $\arg u = \frac{1}{4}\pi$, find the value of a. [2]

Sol (a)

$$\frac{3+2i}{a-5i} \times \frac{a+5i}{a+5i}$$

$$\frac{3a + 15i + 2ai - 10}{a^2+25}$$

$$\frac{3a-10}{a^2+25} + i\,\frac{(2a+15)}{a^2+25}$$

(b)

$$\tan^{-1}\left[\frac{\frac{2a+15}{\cancel{a^2+25}}}{\frac{3a-10}{\cancel{a^2+25}}}\right] = \frac{\pi}{4}$$

$$\frac{2a+15}{3a-10} = \tan\frac{\pi}{4} = 1$$

$$2a+15 = 3a-10$$

$$15+10 = 3a-2a$$

$$\underline{a = 25}$$

Problem : 09709/31/O/N/23/Q5

(a) Given that

$$\sin(x+\tfrac{1}{6}\pi)-\sin(x-\tfrac{1}{6}\pi)=\cos(x+\tfrac{1}{3}\pi)-\cos(x-\tfrac{1}{3}\pi),$$

find the exact value of $\tan x$. [4]

(b) Hence find the exact roots of the equation

$$\sin(x+\tfrac{1}{6}\pi)-\sin(x-\tfrac{1}{6}\pi)=\cos(x+\tfrac{1}{3}\pi)-\cos(x-\tfrac{1}{3}\pi)$$

for $0 \leqslant x \leqslant 2\pi$. [2]

Sol (a) By using compound angle rule.

$$\cancel{\sin x.\cos\frac{\pi}{6}} + \cos x.\sin\frac{\pi}{6} - \cancel{\sin x.\cos\frac{\pi}{6}} + \cos x.\sin\frac{\pi}{6}$$

$$= \cancel{\cos x.\cos\frac{\pi}{3}} - \sin x.\sin\frac{\pi}{3} - \cancel{\cos x.\cos\frac{\pi}{3}} - \sin x.\sin\frac{\pi}{3}$$

$$\cancel{2}\cos x \sin\frac{\pi}{6} = -\cancel{2}\sin x.\sin\frac{\pi}{3}$$

$$\cos x.\frac{1}{\cancel{2}} = -\sin x \frac{\sqrt{3}}{\cancel{2}}$$

$$-\frac{1}{\sqrt{3}} = \frac{\sin x}{\cos x}$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

(b) $\tan x = -\frac{1}{\sqrt{3}}$

Consider $\tan x = \frac{1}{\sqrt{3}}$

$$x = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{6}$$

$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Problem : 09709/31/O/N/23/Q6

The parametric equations of a curve are

$$x = \sqrt{t} + 3, \qquad y = \ln t,$$

for $t > 0$.

(a) Obtain a simplified expression for $\frac{dy}{dx}$ in terms of t. [3]

(b) Hence find the exact coordinates of the point on the curve at which the gradient of the normal is -2. [3]

Sol (a) $\frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}}$ $\qquad \frac{dy}{dt} = \frac{1}{t}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{\frac{t^{-1/2}}{2}} = \frac{2}{t \cdot t^{-1/2}}$$

$$= \frac{2}{\sqrt{t}}$$

(b) $+\frac{\sqrt{t}}{2} = +2$

$\sqrt{t} = 4$

$t = 16$

$x = \sqrt{16} + 3 = 7$

$y = \ln 16$

$(7, \ln 16)$

Problem : 09709/31/O/N/23/Q7

The variables x and θ satisfy the differential equation

$$\frac{x}{\tan\theta}\frac{dx}{d\theta} = x^2 + 3.$$

It is given that $x = 1$ when $\theta = 0$.

Solve the differential equation, obtaining an expression for x^2 in terms of θ. [7]

Sol By variable seperable method

$$\int \frac{x\,dx}{x^2+3} = \int \tan\theta \, d\theta$$

By integration substitution method

$$\frac{1}{2}\ln(x^2+3) = -\ln(\cos\theta) + c$$

let $x^2+3 = u$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$
let $t = \cos\theta$
$dt = -\sin\theta \, d\theta$

$x = 1$ when $\theta = 0$

$$\frac{1}{2}\ln 4 + \ln(1) = c$$

$$\ln 2 = c$$

$$\frac{1}{2}\ln(x^2+3) = -\ln\cos\theta + \ln 2$$

$$\cancel{\ln}(x^2+3)^{1/2} = \cancel{\ln}\frac{2}{\cos\theta}$$

$$x^2+3 = \frac{4}{\cos^2\theta}$$

$$x^2 = \frac{4}{\cos^2\theta} - 3$$

Problem : 09709/31/O/N/23/Q8

(a) By sketching a suitable pair of graphs, show that the equation

$$\sqrt{x} = e^x - 3$$

has only one root. [2]

(b) Show by calculation that this root lies between 1 and 2. [2]

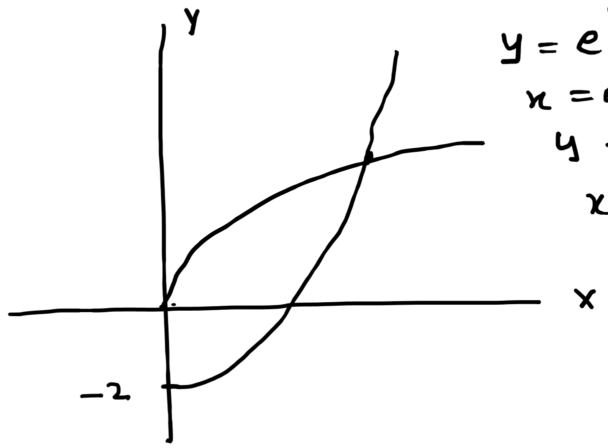
(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln\left(3 + \sqrt{x_n}\right)$$

converges, then it converges to the root of the equation in **(a)**. [1]

(d) Use the iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a)



$y = e^x - 3$

$x = 0 \quad y = -2$

$y = \sqrt{x}$

$x = 0 \quad y = 0$

(b) $\sqrt{x} - e^x + 3$

$x = 1 \quad \sqrt{1} - e + 3 = 1.28$

$x = 2 \quad \sqrt{2} - e^2 + 3 = -2.38$

Since sign of both results are different hence root lies between 1 and 2.

(c) $x = \ln(3 + \sqrt{x})$

$e^x = 3 + \sqrt{x}$

$\sqrt{x} = e^x - 3$

(d)

$$x_{n+1} = \ln(3+\sqrt{x})$$

$x_0 = 1$ $\quad x_1 = \ln(3+\sqrt{1}) = 1.3862$

$$x_2 = \ln(3+\sqrt{x_1}) = 1.4296$$

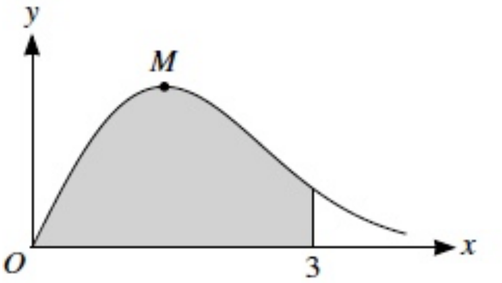
$$x_3 = \ln(3+\sqrt{x_2}) = \underline{1.43}40$$

$$x_4 = \ln(3+\sqrt{x_3}) = \underline{1.43}44$$

$$x_5 = \ln(3+\sqrt{x_4}) = \underline{1.43}45$$

hence root is 1.43

Problem : 09709/31/O/N/23/Q9



The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) Using the substitution $x = \sqrt{u}$, or otherwise, find by integration the exact area of the shaded region bounded by the curve, ~~the x-axis~~ and the line $x = 3$. [5]

Sol (a)

$$y = x \cdot e^{-\frac{1}{4}x^2}$$

$$\frac{dy}{dx} = x \cdot e^{-\frac{1}{4}x^2} \times -\frac{1}{\cancel{4}\,2} \times \cancel{2}x + 1 \cdot e^{-\frac{1}{4}x^2}$$

Since M is stationary point hence $\frac{dy}{dx} = 0$

$$0 = e^{-\frac{1}{4}x^2}\left[-\frac{x^2}{2} + 1\right]$$

$$-\frac{x^2}{2} + 1 = 0$$

$$+\frac{x^2}{2} = +1$$

$$x^2 = 2 \quad \therefore \; x = \pm\sqrt{2} \qquad x = \sqrt{2}$$

$$y = \sqrt{2} \times e^{-\frac{1}{\cancel{4}\,2} \times \cancel{2}}$$

$$= \sqrt{2} \cdot e^{-\frac{1}{2}}$$

hence Coordinate of $M\left(\sqrt{2}, \sqrt{2}e^{-1/2}\right)$

(b)

$$y = x \cdot e^{-\frac{1}{4}x^2}$$

$$\text{Area} = \int_0^3 x \cdot e^{-1/4 x^2}\, dx$$

by integration by Substitution

$x = \sqrt{u}$ $\quad$ $x = 0 \quad u = 0$

$u = x^2$ $\quad$ $x = 3 \quad u = 9$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\frac{1}{2}\int_0^9 e^{-\frac{1}{4}u}\,du = \frac{1}{2}\left[e^{-\frac{1}{4}u}\right]_0^9 \times \frac{1}{-1/4}$$

$$= -\frac{4}{2}\left[e^{-9/4} - 1\right]$$

$$= -2\left[e^{-9/4} - 1\right]$$

$$= 2 - 2e^{-9/4}$$

Problem : 09709/31/O/N/23/Q10

Let $f(x) = \dfrac{24x + 13}{(1 - 2x)(2 + x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Sol (a) $\dfrac{24x+13}{(1-2x)(2+x)^2} = \dfrac{A}{1-2x} + \dfrac{B}{2+x} + \dfrac{C}{(2+x)^2}$

$24x+13 = A(2+x)^2 + B(2+x)(1-2x) + C(1-2x)$

$x = -2$

$24(-2)+13 = 0 + 0 + C(1-2(-2)) = C(5)$

$-35 = 5C \quad \therefore \underline{C = -7}$

$x = \frac{1}{2}$

$24 \times \frac{1}{2} + 13 = A\left(2+\frac{1}{2}\right)^2 + 0 + 0$

$25 = \frac{25}{4}A \quad \therefore \underline{A = 4}$

Consider Const terms

$13 = 4A + 2B + C$

$13 = 16 + 2B - 7$

$\underline{B = 2}$

$\dfrac{4}{(1-2x)} + \dfrac{2}{(2+x)} - \dfrac{7}{(2+x)^2}$

(b) $4(1-2x)^{-1} + 2(2+x)^{-1} - 7(2+x)^{-2}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2$

$4(1-2x)^{-1} = 4(1 + 2x + 4x^2)$

$= 4 + 8x + 16x^2$

$\frac{2}{2}\left(1+\frac{x}{2}\right)^{-1} = 1 - \frac{x}{2} + \frac{x^2}{4}$

$-\frac{7}{4}\left(1+\frac{x}{2}\right)^{-2} = -\frac{7}{4}\left(1 - x + \frac{-2(-2-1)}{2!}\frac{x^2}{4}\right)$

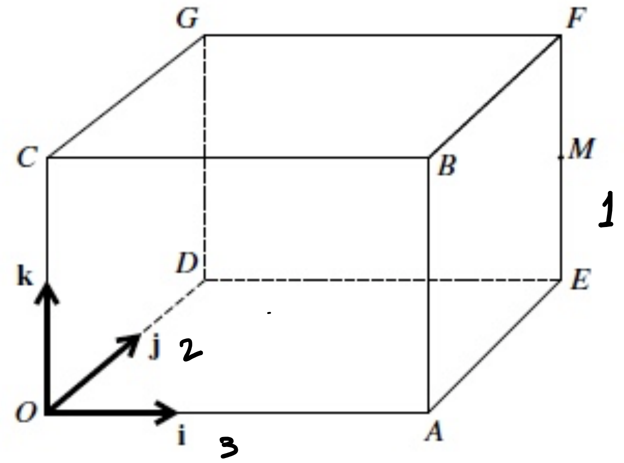
$$= -\frac{7}{4}\left(1 - x + \frac{3}{4}x^2\right)$$

$$= -\frac{7}{4} + \frac{7}{4}x - \frac{21}{16}x^2$$

$$4 + 8x + 16x^2 + 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{7}{4} + \frac{7}{4}x - \frac{21}{16}x^2$$

$$\frac{13}{4} + \frac{37}{4}x + \frac{239}{16}x^2$$

Problem : 09709/31/O/N/23/Q11



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 3$ units, $OC = 2$ units and $OD = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OD and OC respectively. M is the midpoint of EF.

(a) Find the position vector of M. [1]

The position vector of P is $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(b) Calculate angle PAM. [4]

(c) Find the exact length of the perpendicular from P to the line passing through O and M. [5]

Sol (a) $3i + 2j + k$

(b) $\angle PAM$ $A(3,0,0)$ $P(1,1,2)$ $M(3,2,1)$

$\vec{AP} = \vec{OP} - \vec{OA} = -2i + j + 2k$

$AM = \vec{OM} - \vec{OA} = 0i + 2j + k$

$|\vec{AP}| = \sqrt{(-2)^2 + 1^2 + (2)^2} = 3$

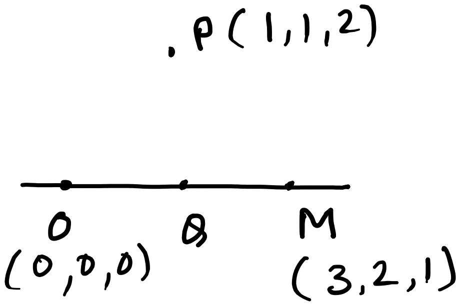
$|\vec{AM}| = \sqrt{0 + 2^2 + 1^2} = \sqrt{5}$

$AP \cdot AM = 0 + 2\times 1 + 2\times 1 = 4$

$\angle PAM = \cos^{-1} \dfrac{4}{3\sqrt{5}} = 53.39$

$= 53.4^\circ$

(c) Eq of line $\overrightarrow{OM}$



$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

So coordinate of Q $(3\lambda, 2\lambda, \lambda)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 3\lambda - 1 \\ 2\lambda - 1 \\ \lambda - 2 \end{pmatrix}$$

$\overrightarrow{PQ} \perp$ direction vector of line O and M

$$3(3\lambda - 1) + 2(2\lambda - 1) + 1(\lambda - 2) = 0$$

$$9\lambda - 3 + 4\lambda - 2 + \lambda - 2 = 0$$

$$14\lambda - 7 = 0$$

$$\lambda = 1/2$$

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \times \frac{1}{2} - 1 \\ 2 \times \frac{1}{2} - 1 \\ \frac{1}{2} - 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -3/2 \end{pmatrix}$$

Exact value of Perpendicular length

$$|\overrightarrow{PQ}| = \sqrt{\left(\frac{1}{2}\right)^2 + 0 + \left(\frac{-3}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{9}{4}}$$

$$= \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$$