Problem : 09709/32/F/M/24/Q1

Find the quotient and remainder when $x^4-3x^3+9x^2-12x+27$ is divided by x^2+5. [3]

Sol

$$x^2+5 \,\big)\, x^4-3x^3+9x^2-12x+27 \,\big(\, x^2-3x+4$$

$$\underset{-}{\cancel{x^4}} \qquad \pm 5x^2$$

$$-3x^3+4x^2-12x$$

$$\underset{+}{-}\cancel{3x^3} \qquad \underset{+}{-}15x$$

$$4x^2+3x+27$$

$$\underset{-}{\cancel{4x^2}} \qquad \pm 20$$

$$3x+7$$

Quotient x^2-3x+4

Remainder $3x+7$

Problem : 09709/32/F/M/24/Q2

(a) Find the coefficient of x^2 in the expansion of $(2x-5)\sqrt{4-x}$. [4]

(b) State the set of values of x for which the expansion in part **(a)** is valid. [1]

Sol (a) $(2x-5)\,4^{1/2}\left(1-\frac{x}{4}\right)^{1/2}$

$$2(2x-5)\left[1-\frac{x}{8}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x}{4}\right)^2\right]$$

$$(4x-10)\left[1-\frac{x}{8}-\frac{1}{128}x^2\right]$$

$$-\frac{x^2}{2}+\frac{10}{128}x^2$$

$$-\frac{1}{2}+\frac{10}{128}$$

$$-\frac{27}{64}$$

(b) $|x|<1$

$$\left|\frac{x}{4}\right|<1$$

$$|x|<4$$

Problem : 09709/32/F/M/24/Q3

It is given that $z = -\sqrt{3} + i$.

(a) Express z^2 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

(b) The complex number ω is such that $z^2\omega$ is real and $\left|\frac{z^2}{\omega}\right| = 12$.

Find the two possible values of ω, giving your answers in the form $Re^{i\alpha}$, where $R > 0$ and $-\pi < \alpha \leqslant \pi$. [3]

Sol (a) $z = -\sqrt{3} + i$

$r = \sqrt{(-\sqrt{3})^2 + 1} = \sqrt{3+1} = 2$

$\theta = \tan^{-1} \frac{1}{-\sqrt{3}} = -\frac{\pi}{6}$

$z = 2e^{-\frac{\pi}{6}i}$

$z^2 = 2^2 e^{-\frac{\pi}{3}i} = 4e^{-\frac{\pi}{3}i}$

(b) $z^2 w = \text{real}$

$\alpha + \theta = -\pi$

$\alpha = -\pi - \theta$

$= -\pi + \frac{\pi}{3}$

$= -\frac{2\pi}{3}$

$\frac{4}{12} e^{-\frac{2}{3}\pi i}$

$\frac{1}{3} e^{-\frac{2}{3}\pi i}$

$\alpha = -\theta$

$= \frac{\pi}{3}$

$\frac{1}{3} e^{\frac{\pi}{3}i}$

Problem : 09709/32/F/M/24/Q4

The positive numbers p and q are such that

$$\ln\left(\frac{p}{q}\right) = a \quad \text{and} \quad \ln\left(q^2 p\right) = b.$$

Express $\ln\left(p^7 q\right)$ in terms of a and b. [4]

Sol

$\ln p - \ln q = a \quad \times 2$

$2\ln q + \ln p = b$

$3\ln p = 2a + b$

$\ln p = \dfrac{2a+b}{3}$

$-3\ln q = a - b$

$\ln q = \dfrac{a-b}{-3}$

$\ln p^7 q$

$7\ln p + \ln q$

$7\left[\dfrac{2a+b}{3}\right] + \left[\dfrac{a-b}{-3}\right]$

$\dfrac{14a + 7b - a + b}{3}$

$\dfrac{13a + 8b}{3}$

Problem : 09709/32/F/M/24/Q5

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-4-2i| \leqslant 3$ and $|z| \geqslant |10-z|$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

Sol (a)

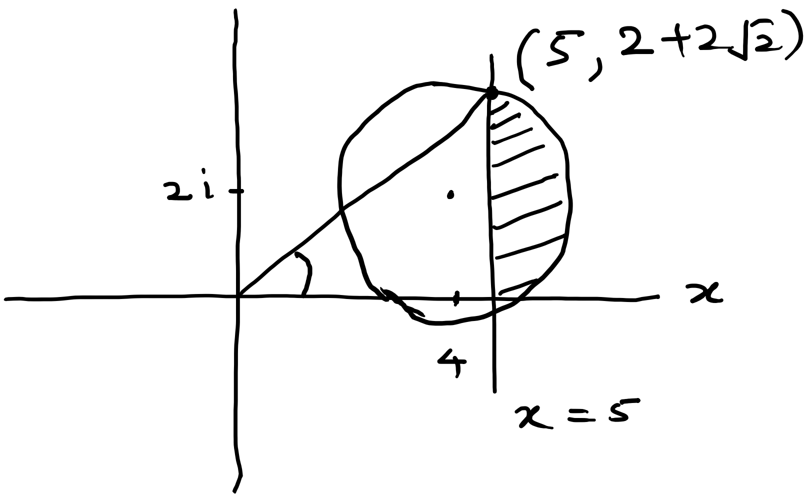
$$(x-4)^2+(y-2)^2 \leq 3^2$$

$$x^2+y^2 \geqslant (10-x)^2+(-y)^2$$

$$\cancel{x^2}+\cancel{y^2} \geqslant 100-20x+\cancel{x^2}+\cancel{y^2}$$

$$20x \geqslant 100$$

$$x \geqslant 5$$



(b)

$$(5-4)^2+(y-2)^2 = 9$$

$$1+(y-2)^2 = 9$$

$$(y-2)^2 = 8$$

$$y = 2+2\sqrt{2}$$

$$\theta = \tan^{-1} \frac{2+2\sqrt{2}}{5}$$

$$= 0.768$$

Problem : 09709/32/F/M/24/Q6

The equation of a curve is $2y^2 + 3xy + x = x^2$.

(a) Show that $\frac{dy}{dx} = \frac{2x - 3y - 1}{4y + 3x}$. [4]

(b) Hence show that the curve does **not** have a tangent that is parallel to the x-axis. [3]

Sol (a) $2y^2 + 3xy + x - x^2 = 0$

$$4y\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) + 1 - 2x = 0$$

$$4y\frac{dy}{dx} + 3x\frac{dy}{dx} = 2x - 3y - 1$$

$$(4y + 3x)\frac{dy}{dx} = 2x - 3y - 1$$

$$\frac{dy}{dx} = \frac{2x - 3y - 1}{4y + 3x}$$

(b) tangent parallel to x-axis

$$\frac{dy}{dx} = 0$$

$$\frac{2x - 3y - 1}{4y + 3x} = 0$$

$$2x - 3y - 1 = 0$$

$$\frac{2x - 1}{3} = y \checkmark$$

$$2y^2 + 3xy + x - x^2 = 0$$

$$2\left(\frac{2x-1}{3}\right)^2 + \cancel{3}x\left(\frac{2x-1}{\cancel{3}}\right) + x - x^2 = 0$$

$$\frac{2(2x-1)^2}{9} + x(2x-1) + x - x^2 = 0$$

$$2(2x-1)^2 + 9x(2x-1) + 9x - 9x^2 = 0$$

$$2(4x^2 - 4x + 1) + 18x^2 - \cancel{9x} + \cancel{9x} - 9x^2 = 0$$

$$8x^2 - 8x + 2 + 18x^2 - 9x^2 = 0$$

$$17x^2 - 8x + 2 = 0$$

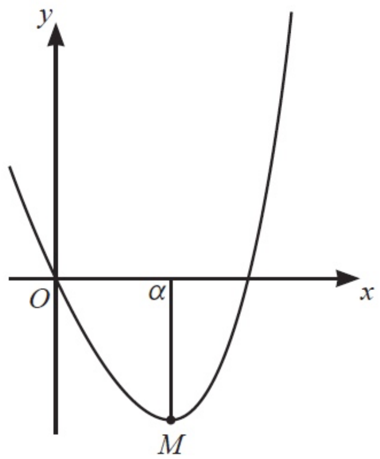
$$b^2 - 4ac$$

$$(-8)^2 - 4 \times 17 \times 2 = 64 - 136 = -72$$

Since discriminant is negative hence no real solution of quadratic eq.

So we can say that curve doesnot have tangent parallel to x-axis.

Problem : 09709/32/F/M/24/Q7



The diagram shows the curve $y = xe^{2x} - 5x$ and its minimum point M, where $x = \alpha$.

(a) Show that α satisfies the equation $\alpha = \frac{1}{2}\ln\left(\frac{5}{1+2\alpha}\right)$. [3]

(b) Verify by calculation that α lies between 0.4 and 0.5 . [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a)

$$y = x \cdot e^{2x} - 5x$$

$$\frac{dy}{dx} = x \cdot e^{2x} \cdot 2 + e^{2x} - 5$$

$$\frac{dy}{dx} = 0 \qquad 0 = e^{2x}(2x+1) - 5$$

$$(2x+1)e^{2x} = 5$$

$$e^{2x} = \frac{5}{2x+1}$$

$$2x = \ln \frac{5}{2x+1}$$

$$x = \frac{1}{2} \ln \frac{5}{2x+1}$$

$$\alpha = \frac{1}{2} \ln \frac{5}{1+2\alpha}$$

(b) $$\alpha - \frac{1}{2} \ln \frac{5}{1+2\alpha}$$

$\alpha = 0.4$ $$0.4 - \frac{1}{2} \ln \frac{5}{1+2(0.4)}$$

$$-0.1108$$

$\alpha = 0.5$ $$0.5 - \frac{1}{2} \ln \frac{5}{1+2(0.5)}$$

$$0.04185$$

Since the values are different due to sign hence α lies between 0.4 and 0.5.

(c) $$\alpha_{n+1} = \frac{1}{2} \ln \frac{5}{1+2\alpha_n}$$

$$\alpha_0 = 0.4$$

$$\alpha_1 = \frac{1}{2} \ln \frac{5}{1+2(0.4)}$$

$$= 0.5108$$

$$\alpha_2 = \frac{1}{2} \ln \frac{5}{1+2(0.5108)}$$

$$= 0.4527$$

$$\alpha_3 = \frac{1}{2} \ln \frac{5}{1+2(0.4527)}$$

$$= 0.4823$$

$$\alpha_4 = \frac{1}{2} \ln \frac{5}{1+2(0.4823)}$$

$$= 0.4670$$

$$\alpha_5 = 0.4749$$

$$\alpha_6 = 0.4708$$

$$\alpha_7 = 0.4729$$

hence α root of equation value

$= 0.47$

Problem : 09709/32/F/M/24/Q8

(a) Express $3\sin x + 2\sqrt{2}\cos\left(x + \frac{1}{4}\pi\right)$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. State the exact value of R and give α correct to 3 decimal places. [4]

(b) Hence solve the equation

$$6\sin\tfrac{1}{2}\theta + 4\sqrt{2}\cos\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right) = 3$$

for $-4\pi < \theta < 4\pi$. [5]

Sol (a) $3\sin x + 2\sqrt{2}\left[\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right]$

$3\sin x + 2\cos x - 2\sin x$

$\sin x + 2\cos x = R\sin(x+\alpha)$

$R = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\alpha = \tan^{-1}\frac{2}{1} = 1.107$

(b) $6\sin\frac{1}{2}\theta + 4\sqrt{2}\cos\left(\frac{1}{2}\theta + \frac{\pi}{4}\right) = 3$

$3\sin\frac{\theta}{2} + 2\sqrt{2}\cos\left(\frac{1}{2}\theta + \frac{\pi}{4}\right) = 1.5$

$\sqrt{5}\sin\left(\frac{1}{2}\theta + 1.107\right) = 1.5$

$\sin\left(\frac{1}{2}\theta + 1.107\right) = \frac{1.5}{\sqrt{5}}$

$\frac{1}{2}\theta + 1.107 = \sin^{-1}\left(\frac{1.5}{\sqrt{5}}\right)$

$= 0.7353$

$\theta = (0.7353 - 1.107)\times 2$

$= -0.74$

$$\frac{1}{2}\theta + 1.107 = \pi - \sin^{-1}\left(\frac{1.5}{\sqrt{5}}\right)$$

$$\theta = \left(\pi - \sin^{-1}\left(\frac{1.5}{\sqrt{5}}\right) - 1.107\right) \times 2$$

$$= 2.598$$

$$= 2.60$$

Problem : 09709/32/F/M/24/Q9

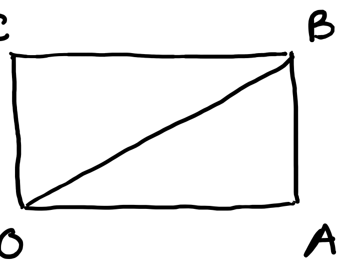
Relative to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 8\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}.$$

(a) Show that $OABC$ is a rectangle. [4]

(b) Use a scalar product to find the acute angle between the diagonals of $OABC$. [4]

Sol (a)



$\overrightarrow{OB} = 8i + 2j - 6k$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 5i - 2j + k$
$\quad - 3i - 4j + 7k$

$= 2i - 6j + 8k$

$|\overrightarrow{OB}| = \sqrt{8^2 + 2^2 + (-6)^2} = \sqrt{64 + 4 + 36}$
$= \sqrt{104}$

$|\overrightarrow{CA}| = \sqrt{2^2 + (-6)^2 + (8)^2} = \sqrt{4 + 36 + 64}$
$= \sqrt{104}$

hence OABC is a rectangle as length of both diagonal is same.

(b)

$$\cos\theta = \frac{\overrightarrow{OB} \cdot \overrightarrow{CA}}{|OB||\overrightarrow{CA}|}$$

$$= \frac{8 \times 2 - 2 \times 6 - 6 \times 8}{\sqrt{104} \times \sqrt{104}}$$

$$\theta = \cos^{-1}\left[\frac{16 - 12 - 48}{104}\right]$$

$\theta = 115.0^\circ$

Acute angle $= 180 - 115 = 65^\circ$

Problem : 09709/32/F/M/24/Q10

Let $f(x) = \dfrac{36a^2}{(2a+x)(2a-x)(5a-2x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_{-a}^{a} f(x)\,dx$, giving your answer in the form $p\ln q + r\ln s$ where p and r are integers and q and s are prime numbers. [5]

Sol (a) $\dfrac{36a^2}{(2a+x)(2a-x)(5a-2x)} = \dfrac{A}{(2a+x)} + \dfrac{B}{(2a-x)} + \dfrac{C}{(5a-2x)}$

$$36a^2 = A(2a-x)(5a-2x) + B(2a+x)(5a-2x) + C(2a+x)(2a-x)$$

$x = 2a$

$$36\cancel{a^2} = B(4a)(5a-4a) = B\,4\cancel{a^2}$$

$$B = 9$$

$x = -2a$

$$36a^2 = A(4a)(5a+4a)$$

$$36\cancel{a^2} = A(4\cancel{a})(9\cancel{a})$$

$$A = 1$$

$2x = 5a$

$x = \dfrac{5a}{2}$

$$36a^2 = C\left(2a - \frac{5a}{2}\right)\left(2a + \frac{5a}{2}\right)$$

$$36\cancel{a^2} = C\left(-\frac{\cancel{a}}{2}\right)\left(\frac{9\cancel{a}}{2}\right)$$

$$36 = C \; -\frac{9}{4}$$

$$C = -16$$

(b)

$$\int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{a} \left[\frac{1}{(2a+x)} + \frac{9}{(2a-x)} - \frac{16}{(5a-2x)} \right] dx$$

$$= \left[\ln|2a+x| + 9\,\frac{\ln|2a-x|}{-1} - 16\,\frac{\ln|5a-2x|}{-2} \right]_{-a}^{a}$$

$$= \left[\ln|2a+x| - 9\ln|2a-x| + 8\ln|5a-2x| \right]_{-a}^{a}$$

$$= \ln|3a| - 9\ln|a| + 8\ln|3a| - \ln|a|$$
$$+ 9\ln|3a| - 8\ln|7a|$$

$$= \ln|3| + \cancel{\ln|a|} - 9\cancel{\ln|a|} + 8\ln|3| + 8\cancel{\ln|a|}$$
$$- \cancel{\ln|a|} + 9\ln|3| + 9\cancel{\ln|a|} - 8\ln|7| - 8\cancel{\ln|a|}$$

$$= 18\ln|3| - 8\ln|7|$$

Problem : 09709/32/F/M/24/Q11

The variables y and θ satisfy the differential equation

$$(1+y)(1+\cos 2\theta)\frac{dy}{d\theta} = e^{3y}.$$

It is given that $y = 0$ when $\theta = \frac{1}{4}\pi$.

Solve the differential equation and find the exact value of $\tan\theta$ when $y = 1$. [9]

Sol By variable seperable method.

$$\int (1+y)e^{-3y}\,dy = \int \frac{1}{1+\cos 2\theta}\,d\theta$$

$\int uv' = uv - \int u'v$

$1+\cos 2\theta$
$= \cancel{1} + 2\cos^2\theta \cancel{-1}$

$u = 1+y \quad v' = e^{-3y}$

$= \frac{1}{2}\int \sec^2\theta\, d\theta$

$u' = 1 \quad v = -\frac{1}{3}e^{-3y}$

$(1+y)\left(-\frac{1}{3}e^{-3y}\right) - \int 1 \cdot -\frac{1}{3}e^{-3y}\,dy = \frac{1}{2}\tan\theta$

$-\frac{1}{3}e^{-3y}(1+y) + \frac{1}{3}\int e^{-3y}\,dy$

$-\frac{1}{3}e^{-3y}(1+y) - \frac{1}{9}e^{-3y} + c$

$-\frac{1}{3}e^{-3y}(1+y) - \frac{1}{9}e^{-3y} + c = \frac{1}{2}\tan\theta$

$y = 0 \quad \theta = \pi/4$

$-\frac{1}{3} - \frac{1}{9} + c = \frac{1}{2}\tan\frac{\pi}{4} = \frac{1}{2}$

$c = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$

$-\frac{1}{3}e^{-3y}(1+y) - \frac{1}{9}e^{-3y} + \frac{17}{18} = \frac{1}{2}\tan\theta$

Substitute $y = 1$ in particular solution

$$\frac{1}{2}\tan\theta = -\frac{1}{3}e^{-3}(2) - \frac{1}{9}e^{-3} + \frac{17}{18}$$

$$= -\frac{1}{3}e^{-3}\left(2 + \frac{1}{3}\right) + \frac{17}{18}$$

$$\frac{1}{2}\tan\theta = -\frac{7}{9}e^{-3} + \frac{17}{18}$$

$$\tan\theta = -\frac{14}{9}e^{-3} + \frac{17}{9}$$

$$= \frac{17}{9} - \frac{14}{9}e^{-3}$$