Problem : 09709/32/O/N/23/Q1

(a) Sketch the graph of $y = |4x - 2|$. [1]

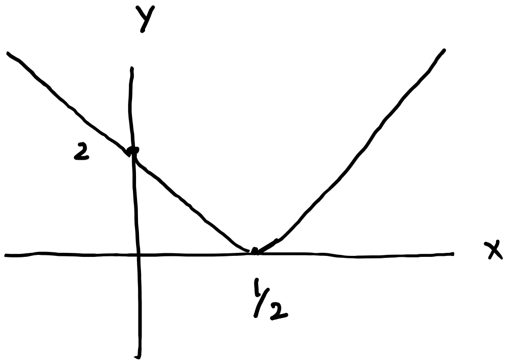
(b) Solve the inequality $1 + 3x < |4x - 2|$. [4]

Sol (a) $4x - 2 = 0$

$x = \frac{1}{2}$

$y = |4 \times 0 - 2|$

$= 2$



(b) $1 + 3x < |4x - 2|$

By squaring both sides

$(1+3x)^2 < (4x-2)^2$

$1 + 6x + 9x^2 < 16x^2 + 4 - 16x$

$7x^2 - 22x + 3 > 0$

$7x^2 - 21x - x + 3 > 0$

$7x(x-3) - 1(x-3) > 0$

$(7x-1)(x-3) > 0$

$x < \frac{1}{7} \cup x > 3$

Problem : 09709/32/O/N/23/Q2

The parametric equations of a curve are

$$x = (\ln t)^2, \qquad y = e^{2-t^2},$$

for $t > 0$.

Find the gradient of the curve at the point where $t = e$, simplifying your answer. [4]

Sol

$$\frac{dx}{dt} = 2\frac{\ln t}{t} \qquad \frac{dy}{dt} = e^{2-t^2} \times -2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\cancel{2}t^2 e^{2-t^2}}{\cancel{2}\ln t}$$

$t = e$

$$\frac{dy}{dx} = \frac{-e^2 . e^{2-e^2}}{\ln e}$$

$$= -e^{4-e^2} \qquad \text{where } \ln e = 1$$

Problem : 09709/32/O/N/23/Q3

The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12.

Find the values of a and b. [5]

Sol

$$P(x) = 2x^3 + ax^2 - 11x + b$$

$x = \frac{1}{2}$

$$P\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right)^3 + a\left(\tfrac{1}{2}\right)^2 - 11\left(\tfrac{1}{2}\right) + b$$

$$0 = \frac{1}{4} + \frac{a}{4} - \frac{11}{2} + b$$

$$0 = 1 + a - 22 + 4b$$

$$a + 4b = 21 \quad \text{———} \quad (i)$$

$x = -1$

$$P(-1) = 2(-1)^3 + a(-1)^2 - 11(-1) + b$$

$$12 = -2 + a + 11 + b$$

$$a + b = 3 \quad \text{———} \quad (ii)$$

by solving eq (i) and (ii)

$$a = -3 \quad b = 6$$

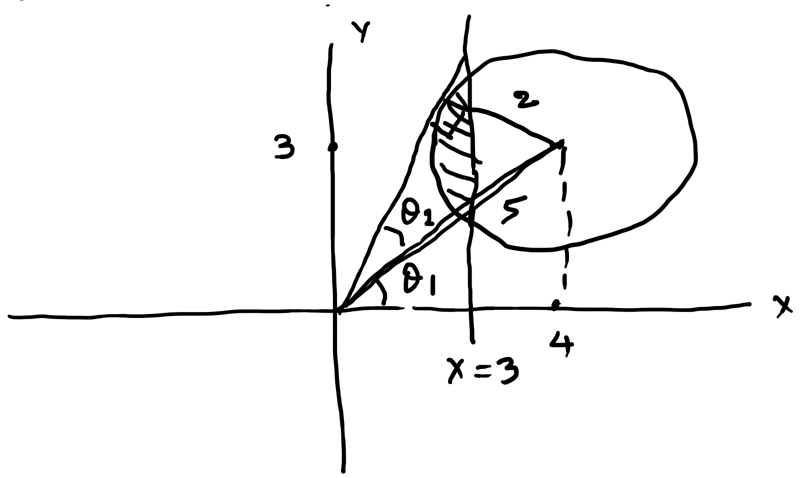
Problem : 09709/32/O/N/23/Q4

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 4 - 3i| \leqslant 2$ and $\text{Re}\, z \leqslant 3$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

Sol (a) $|z - 4 - 3i| \leq 2$ $\quad$ $\text{Re}(z) \leq 3$

$(x-4)^2 + (y-3)^2 \leq 2$ $\quad$ $x \leq 3$



(b) Greatest $\text{Arg}(z) = \theta_1 + \theta_2$

$\theta_1 = \tan^{-1}\dfrac{3}{4}$ $\quad$ $\theta_2 = \sin^{-1}\dfrac{2}{5}$

$= \tan^{-1}\dfrac{3}{4} + \sin^{-1}\dfrac{2}{5}$

Problem : 09709/32/O/N/23/Q5

Find the exact value of $\displaystyle\int_0^6 \frac{x(x+1)}{x^2+4}\,dx$. [6]

Sol

$$\int_0^6 \frac{x^2+x}{x^2+4}\,dx$$

$$x^2+4 \overline{)\, \cancel{x^2}+x \,}\ (1$$
$$\underline{-\cancel{x^2}\pm 4}$$
$$x-4$$

$$\int_0^6 \left[1+\frac{x-4}{x^2+4}\right]dx$$

let $u = x^2+4$

$du = 2x\,dx$

$x\,dx = \frac{du}{2}$

$\tan^{-1}x = \int \frac{1}{1+x^2}$

$$\int_0^6 \left[1+\frac{x}{x^2+4}-\frac{4}{4+x^2}\right]dx$$

$$\left[x+\frac{1}{2}\ln(x^2+4)-2\tan^{-1}\frac{x}{2}\right]_0^6$$

$$\left[6+\frac{1}{2}\ln 40-2\tan^{-1}3\right]-\left[0+\frac{1}{2}\ln 4-0\right]$$

$$6+\frac{1}{2}\ln 10-2\tan^{-1}3$$

Problem : 09709/32/O/N/23/Q6

(a) By sketching a suitable pair of graphs, show that the equation

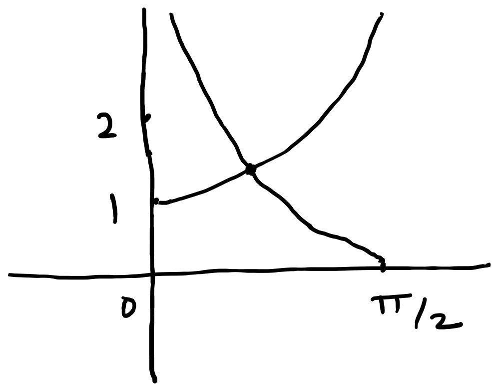
$$\cot x = 2 - \cos x$$

has one root in the interval $0 < x \leqslant \frac{1}{2}\pi$. [2]

(b) Show by calculation that this root lies between 0.6 and 0.8. [2]

(c) Use the iterative formula $x_{n+1} = \tan^{-1}\left(\frac{1}{2 - \cos x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a)



(b) $\cot x - 2 + \cos x$

$x = 0.6 \quad \cot(0.6) - 2 + \cos(0.6)$

$= 0.2870$

$x = 0.8 \quad \cot(0.8) - 2 + \cos(0.8)$

$= -0.3320$

Since sign of result different hence one root will lies between 0.6 and 0.8.

(c) $x_{n+1} = \tan^{-1}\left[\frac{1}{2 - \cos x_n}\right]$

$x_0 = 0.6 \quad x_1 = \tan^{-1}\left[\frac{1}{2 - \cos(0.6)}\right] = 0.7052$

$x_2 = \tan^{-1}\left[\frac{1}{2 - \cos(0.7052)}\right] = 0.6792$

$x_3 = 0.\underline{68}58 \quad x_4 = 0.\underline{68}41$

$x_5 = \underline{0.68}45$

hence root of equation will be $\underline{\underline{0.68}}$

Problem : 09709/32/O/N/23/Q7

(a) By expressing 3θ as $2\theta + \theta$, prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [3]

(b) Hence solve the equation

$$\cos 3\theta + \cos\theta\cos 2\theta = \cos^2\theta$$

for $0° \leqslant \theta \leqslant 180°$. [5]

Sol (a) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta . \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta).\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

(b) $4\cos^3\theta - 3\cos\theta + \cos\theta\underline{\cos 2\theta} = \cos^2\theta$

$4\cos^3\theta - 3\cos\theta + 2\cos^3\theta - \cos\theta = \cos^2\theta$

$6\cos^3\theta - 4\cos\theta - \cos^2\theta = 0$

$\cos\theta = 0$ $\quad$ $6\cos^2\theta - 4 - \cos\theta = 0$

$\theta = 90°$ $\quad$ $\cos\theta = 0.904$ $\quad$ $\cos\theta = -.737$

$\theta = 25.3°$ $\quad$ $\theta = 42.52$

$\theta = 137.47$

$= 137.5°$

Problem : 09709/32/O/N/23/Q8

It is given that $\dfrac{2+3a\mathrm{i}}{a+2\mathrm{i}} = \lambda(2-\mathrm{i})$, where a and λ are real constants.

(a) Show that $3a^2+4a-4=0$. [4]

(b) Hence find the possible values of a and the corresponding values of λ. [3]

Sol (a)

$$\frac{(2+3ai)(a-2i)}{(a+2i)(a-2i)} = \lambda(2-i)$$

$$\frac{2a-4i+3a^2i+6a}{a^2+4} = \lambda(2-i)$$

$$8a+i(3a^2-4) = 2\lambda a^2+8\lambda - i(a^2+4)\lambda$$

$$8a = (2a^2+8)\lambda \qquad 3a^2-4 = -(a^2+4)\lambda$$

$$\lambda = \frac{8a}{2a^2+8} \qquad \lambda = -\frac{(3a^2-4)}{a^2+4}$$

$$\frac{8a}{2a^2+8} = \frac{-3a^2+4}{a^2+4}$$

$$\frac{4a}{\cancel{a^2+4}} = \frac{-3a^2+4}{\cancel{a^2+4}}$$

$$3a^2+4a-4=0$$

(b)

$$3a^2+4a-4=0$$

$$a=\frac{2}{3} \qquad a=-2$$

$$\lambda = \frac{4a}{a^2+4}$$

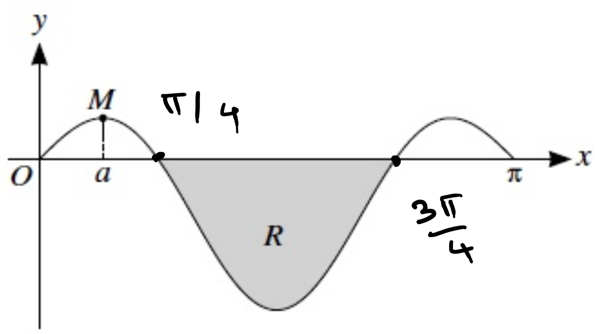
$$a=-2 \quad \lambda = \frac{-8}{8} = -1$$

$a = \frac{2}{3} \qquad \lambda = \dfrac{4 \times \frac{2}{3}}{\left(\frac{2}{3}\right)^2 + 4} = \dfrac{\frac{8}{3}}{\frac{4}{9} + 4} = \dfrac{\frac{8}{3}}{\frac{40}{9}}$

$= \frac{\cancel{8}}{\cancel{3}} \times \frac{\cancel{9}^{3}}{\cancel{40}_{5}} = \frac{3}{5}$

$a = \frac{2}{3} \quad \lambda = 3/5$

Problem : 09709/32/O/N/23/Q9



The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Find the value of a correct to 2 decimal places. [5]

(b) Find the exact area of the region R, giving your answer in simplified form. [4]

Sol (a)

$$\frac{dy}{dx} = \sin x . -\sin 2x . 2 + \cos x \cos 2x$$

$$\frac{dy}{dx} = -2 \sin x \sin 2x + \cos x \cos 2x$$

$$0 = -2 \sin x . 2 \sin x \cos x + \cos x (1 - 2\sin^2 x)$$

$$= \cos x \left[-4 \sin^2 x + 1 - 2 \sin^2 x \right]$$

$$0 = \cos x \left[1 - 6 \sin^2 x \right]$$

$\cos x = 0$

$x = \frac{\pi}{2}$

$1 - 6 \sin^2 x = 0$

$6 \sin^2 x = 1$

$\sin x = \frac{1}{\sqrt{6}}$

$x = \sin^{-1} \frac{1}{\sqrt{6}}$

$= 0.42$

(b) $\sin x . \cos 2x = 0$

$\sin x = 0 \qquad \cos 2x = 0$

$x = 0, \pi \qquad 2x = \cos^{-1}(0)$

$2x = \frac{\pi}{2} \qquad 2x = \frac{3}{2}\pi$

$x = \frac{\pi}{4} \qquad x = \frac{3\pi}{4}$

$\int_{\pi/4}^{3\pi/4} \sin x . \cos 2x \, dx \qquad \cos 2x = 2\cos^2 x - 1$

$\int_{\pi/4}^{3\pi/4} (\sin x . 2\cos^2 x - \sin x) dx$

$2\int_{\pi/4}^{3\pi/4} \sin x \cos^2 x \, dx - \int_{\pi/4}^{3\pi/4} \sin x \, dx$

$-\left[-\cos x\right]_{\pi/4}^{\frac{3\pi}{4}}$

$\left[\cos \frac{3\pi}{4} - \cos \frac{\pi}{4}\right]$

$+(-\sqrt{2})$

let $\cos x = t$

$-\sin x dx = dt$

Now limit will change

$x = \pi/4 \quad t = \frac{1}{\sqrt{2}}$

$x = \frac{3\pi}{4} \quad t = -\frac{1}{\sqrt{2}}$

$-2\int_{1/\sqrt{2}}^{-\frac{1}{\sqrt{2}}} t^2 dt$

$-2\left[\frac{t^3}{3}\right]_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}}$

$-\frac{2}{3}\left[-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]$

$-\frac{2}{3} \times -\frac{1}{\sqrt{2}}$

$\frac{2}{3\sqrt{2}} - \sqrt{2} = \frac{2-6}{3\sqrt{2}} = \frac{-4}{3\sqrt{2}}$

$= \frac{2\sqrt{2}}{3}$ ✓

Problem : 09709/32/O/N/23/Q10

The equations of the lines l and m are given by

$$l:\ \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad m:\ \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix},$$

where c is a positive constant. It is given that the angle between l and m is $60°$.

(a) Find the value of c. [4]

(b) Show that the length of the perpendicular from $(6, -3, 6)$ to l is $\sqrt{11}$. [5]

Sol (a)

$$\cos 60 = \frac{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix}}{\sqrt{6}\ \sqrt{4+16+c^2}}$$

$$\frac{1}{2} = \frac{-2+4+2c}{\sqrt{6}\ \sqrt{20+c^2}}$$

$$\sqrt{6}\ \sqrt{20+c^2} = 4+4c$$

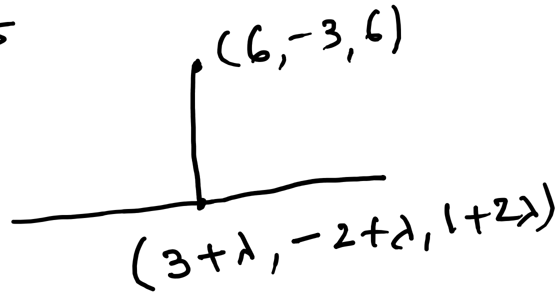
$$6 \times (20+c^2) = (4+4c)^2$$

$$120 + 6c^2 = 16 + 32c + 16c^2$$

$$10c^2 + 32c - 104 = 0$$

$$\underline{\underline{c = 2}} \qquad c = -\frac{26}{5}$$

(b) $l:\ r = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$



$$\begin{pmatrix} 6-3-\lambda \\ -3+2-\lambda \\ 6-1-2\lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda \\ -1-\lambda \\ 5-2\lambda \end{pmatrix}$$

$$1.(3-\lambda) + 1(-1-\lambda) + 2(5-2\lambda) = 0 \qquad \therefore\ \lambda = 2$$

$$3-\lambda - 1 - \lambda + 10 - 4\lambda = 0$$

$$12 - 6\lambda = 0$$

$$\left\| \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2 + (-3)^2 + (1)^2}$$

$$= \sqrt{11}$$

Problem : 09709/32/O/N/23/Q11

The variables x and y satisfy the differential equation

$$x^2\frac{dy}{dx} + y^2 + y = 0.$$

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for y in terms of x. [8]

(b) State what happens to the value of y when x tends to infinity. Give your answer in an exact form. [1]

Sol (a) By variable seperable method

$$x^2\frac{dy}{dx} = -y^2 - y$$

$$\int \frac{dy}{y^2+y} = -\int x^{-2}\,dx$$

$$\int \frac{dy}{y(y+1)} = -\int x^{-2}\,dx$$

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$1 = A(y+1) + B(y)$$

$y = 0 \quad A = 1$

$y = -1 \quad B = -1$

$$\int \frac{1}{y}\,dy - \int \frac{1}{y+1}\,dy = -\int x^{-2}\,dx$$

$$\ln y - \ln y+1 = -\frac{x^{-1}}{-1} + c$$

$$\ln \frac{y}{y+1} = \frac{1}{x} + c$$

$x = 1$ when $y = 1$

$$\ln \frac{1}{2} = 1 + c$$

$$c = \ln \frac{1}{2} - 1 = \ln 1 - \ln 2 - 1$$

$$= -\ln 2 - 1$$

$$\ln \frac{y}{y+1} = \frac{1}{x} - \ln 2 - 1$$

$$\ln \frac{y}{y+1} + \ln 2 = \frac{1-x}{x}$$

$$\ln \frac{2y}{y+1} = \frac{1-x}{x}$$

$$\frac{2y}{y+1} = e^{\frac{1-x}{x}}$$

$$2y = y e^{\frac{1-x}{x}} + e^{\frac{1-x}{x}}$$

$$y\left(2 - e^{\frac{1-x}{x}}\right) = e^{\frac{1-x}{x}}$$

$$y = \frac{e^{\frac{1-x}{x}}}{2 - e^{\frac{1-x}{x}}}$$

(b) $x \to \infty$

$$y = \frac{e^{\frac{1}{\infty}-1}}{2 - e^{\frac{1}{\infty}-1}} = \frac{e^{-1}}{2 - e^{-1}}$$

$$= \frac{\frac{1}{e}}{2 - \frac{1}{e}} = \frac{1}{2e-1}$$