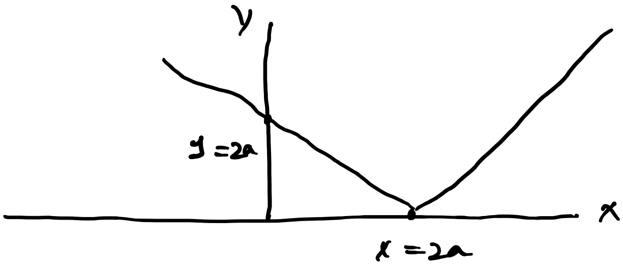
Problem : 09709/32/M/J24/Q1

(a) Sketch the graph of $y = |x - 2a|$, where a is a positive constant. [1]

(b) Solve the inequality $2x - 3a < |x - 2a|$. [2]

Sol (a) $x = 0 \quad y = 2a$

$y = 0 \quad x = 2a$



(b) $2x - 3a < |x - 2a|$

$(2x-3a)^2 < (x-2a)^2$

$4x^2 - 12xa + 9a^2 < x^2 - 4ax + 4a^2$

$3x^2 - 8xa + 5a^2 < 0$

$3x^2 - 5ax - 3ax + 5a^2 < 0$

$x(3x - 5a) - a(3x - 5a) < 0$

$(3x - 5a)(x - a) < 0$

$x = \frac{5}{3}a \qquad x = a$

$x < \frac{5}{3}a$

Problem : 09709/32/M/J24/Q2

Express $\frac{6x^2-9x-16}{2x^2-5x-12}$ in partial fractions. [5]

Sol

$$2x^2-5x-12 \overline{\big)\, 6x^2-9x-16\,}\; 3$$
$$\underline{-\,(6x^2-15x-36)}$$
$$6x+20$$

$$2x^2-5x-12$$
$$2x^2-8x+3x-12$$
$$2x(x-4)+3(x-4)$$
$$(x-4)(2x+3)$$

$$3+\frac{6x+20}{(2x+3)(x-4)}$$

$$\frac{6x+20}{(2x+3)(x-4)}=\frac{A}{2x+3}+\frac{B}{x-4}$$

$$6x+20=A(x-4)+B(2x+3)$$

$x=4$
$$44=A(0)+B(11)$$
$$B=4$$

$x=-3/2$
$$-9+20=A\left(-\frac{3}{2}-4\right)+B(0)$$
$$11=-\frac{11}{2}A$$
$$A=-2$$

$$3+\frac{-2}{2x+3}+\frac{4}{x-4}$$

Problem : 09709/32/M/J24/Q3

The variables x and y satisfy the equation $a^{2y-1} = b^{x-y}$, where a and b are constants.

(a) Show that the graph of y against x is a straight line. [3]

(b) Given that $a = b^3$, state the equation of the straight line in the form $y = px + q$, where p and q are rational numbers in their simplest form. [2]

Sol (a) $(2y-1)\ln a = (x-y)\ln b$

$2y\ln a - \ln a = x\ln b - y\ln b$

$y(2\ln a + \ln b) = x\ln b + \ln a$

✓ $y = \dfrac{\ln b}{\ln a^2 b} x + \dfrac{\ln a}{\ln a^2 b}$ ✓ ✓

$m = \text{gradient} = \dfrac{\ln b}{\ln a^2 b}$

$c = y\text{-intercept} = \dfrac{\ln a}{\ln a^2 b}$

(b) Given that $a = b^3$

Now subsititute

$y = \dfrac{\ln b}{\ln b^7} x + \dfrac{3\ln b}{\ln b^7}$

$= \dfrac{\cancel{\ln b}}{7\cancel{\ln b}} x + \dfrac{3\cancel{\ln b}}{7\cancel{\ln b}}$

$y = \dfrac{1}{7}x + \dfrac{3}{7}$

Problem : 09709/32/M/J24/Q4

The equation of a curve is $ye^{2x} + y^2e^x = 6$. ✓

Find the gradient of the curve at the point where $y = 1$. [6]

Sol

$y = 1 \quad 1.e^{2x} + 1^2e^x = 6$

$e^{2x} + e^x - 6 = 0$

let $e^x = t$

$t^2 + t - 6 = 0$

$t^2 + 3t - 2t - 6 = 0$

$t(t+3) - 2(t+3) = 0$

$(t-2)(t+3) = 0$

$\underline{\underline{e^x = 2}} \quad e^x = -3 \times$

$x = \ln 2$

$ye^{2x} + y^2e^x = 6$

$\frac{dy}{dx} \cdot e^{2x} + ye^{2x} \cdot 2 + 2y\frac{dy}{dx}e^x + y^2 \cdot e^x = 0$

$\underline{\frac{dy}{dx} \times 4} + 8 + 2\underline{\frac{dy}{dx}2} + 2 = 0$

$8\frac{dy}{dx} = -10$

$\frac{dy}{dx} = -\frac{10}{8} = -\frac{5}{4}$

Problem : 09709/32/M/J24/Q5

(a) It is given that the equation $e^{2x} = 5 + \cos 3x$ has only one root.

Show by calculation that this root lies in the interval $0.7 < x < 0.8$. [2]

(b) Show that if a sequence of values in the interval $0.7 < x < 0.8$ given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(5 + \cos 3x_n\right)$$

converges then it converges to the root of the equation in part **(a)**. [1]

(c) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Sol (a) $e^{2x} - 5 - \cos 3x$

$x = 0.7$

-0.4399

$x = 0.8$

0.6904

Since the sign is different hence root lies between 0.7 and 0.8

(b) $e^{2x} = 5 + \cos 3x$

$2x = \ln(5 + \cos 3x)$

$x = \frac{1}{2}\ln(5 + \cos 3x)$

$x_{n+1} = \frac{1}{2}\ln(5 + \cos 3x_n)$ ✓

(c) $x_0 = 0.7 \quad x_1 = \frac{1}{2}\ln(5 + \cos 3(0.7))$

$= 0.75149$

$x_2 = \frac{1}{2}\ln(5 + \cos 3(x_1))$

$= 0.73719$

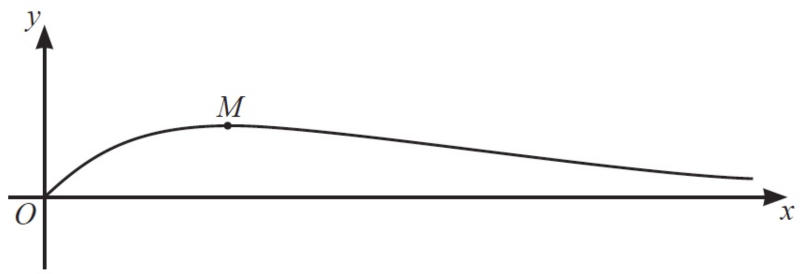
$x_3 = \frac{1}{2}\ln(5 + \cos 3(x_2)) = 0.74105$

$x_4 = 0.73999 \quad x_5 = 0.74028$ ✓

$x_6 = 0.74020$ ✓ $x_7 = 0.74022$ ✓

Root $= 0.740$

Problem : 09709/32/M/J24/Q6



The diagram shows the curve $y = xe^{-ax}$, where a is a positive constant, and its maximum point M. $\frac{dy}{dx} = 0$

(a) Find the exact coordinates of M. [4]

(b) Find the exact value of $\int_0^{\frac{2}{a}} xe^{-ax}\, dx$. [5]

Sol (a) $\frac{dy}{dx} = x \cdot e^{-ax} \cdot (-a) + 1 \cdot e^{-ax}$

$0 = e^{-ax}(-ax+1)$

$e^{-ax} = 0$ ✗ $\quad -ax+1=0 \quad x = \frac{1}{a}$

$y = \frac{1}{a} \cdot e^{-\cancel{a} \times \frac{1}{\cancel{a}}} = \frac{1}{a} e^{-1} = \frac{1}{ae}$

$M\left(\frac{1}{a}, \frac{1}{ae}\right)$

(b) $\int_0^{\frac{2}{a}} x \cdot e^{-ax}\, dx$

By integration By Parts method

$\left[x \cdot \frac{e^{-ax}}{-a}\right]_0^{2/a} - \int_0^{2/a} 1 \cdot \frac{e^{-ax}}{-a}\, dx$

$\left[\frac{x \cdot e^{-ax}}{-a}\right]_0^{2/a} + \frac{1}{a}\left[\frac{e^{-ax}}{-a}\right]_0^{2/a}$

$\left[\frac{x \cdot e^{-ax}}{-a}\right]_0^{2/a} - \frac{1}{a^2}\left[e^{-ax}\right]_0^{2/a}$

$-\frac{1}{a}\left[\frac{2}{a} \cdot e^{-\cancel{a} \times \frac{2}{\cancel{a}}} - 0\right] - \frac{1}{a^2}\left[e^{-\cancel{a} \times \frac{2}{\cancel{a}}} - 1\right]$

$$-\frac{2}{a^2}e^{-2}-\frac{e^{-2}}{a^2}+\frac{1}{a^2}$$

$$-\frac{3e^{-2}}{a^2}+\frac{1}{a^2}$$

$$\frac{1}{a^2}(1-3e^{-2})$$

Problem : 09709/32/M/J24/Q7

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$. [3]

(b) Hence find the exact value of $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \left(\underline{\cos^4\theta - \sin^4\theta} + \underline{4\sin^2\theta\cos^2\theta}\right) d\theta$. [6]

$(2\sin\theta\cos\theta)^2$

Sol (a) $(\cos^2\theta)^2 - (\sin^2\theta)^2$

$\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} \underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta}$

$\cos 2\theta$ (RHS)

(b) $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} (\cos 2\theta + \sin^2 2\theta)\, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 4\theta = 1 - 2\sin^2 2\theta$

$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \cos 2\theta\, d\theta + \int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \frac{1-\cos 4\theta}{2}\, d\theta$

$\left[\frac{\sin 2\theta}{2}\right]_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} + \int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \frac{1}{2}\, d\theta - \int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \frac{\cos 4\theta}{2}\, d\theta$

$\frac{1}{2}\left[\sin \cancel{2} \times \frac{1}{\cancel{8}4}\pi - \sin \cancel{2} \times -\frac{1}{\cancel{8}4}\pi\right] + \frac{1}{2}\left[\frac{1}{8}\pi - \left(-\frac{1}{8}\pi\right)\right]$

$- \frac{1}{2}\left[\frac{\sin 4\theta}{4}\right]_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}$

$\frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] + \frac{\pi}{8} - \frac{1}{8}\left[\sin\frac{\pi}{2} - \sin -\frac{\pi}{2}\right]$

$\frac{1}{\sqrt{2}} + \frac{\pi}{8} - \frac{1}{8}[2]$

$\frac{\sqrt{2}}{2} + \frac{\pi}{8} - \frac{1}{4}$

Problem : 09709/32/M/J24/Q8

The points A, B and C have position vectors $\overrightarrow{OA} = -2\mathbf{i}+\mathbf{j}+4\mathbf{k}$, $\overrightarrow{OB} = 5\mathbf{i}+2\mathbf{j}$ and $\overrightarrow{OC} = 8\mathbf{i}+5\mathbf{j}-3\mathbf{k}$, where O is the origin. The line l_1 passes through B and C.

(a) Find a vector equation for l_1. [3]

The line l_2 has equation $\mathbf{r} = -2\mathbf{i}+\mathbf{j}+4\mathbf{k}+\mu(3\mathbf{i}+\mathbf{j}-2\mathbf{k})$.

(b) Find the coordinates of the point of intersection of l_1 and l_2. [4]

(c) The point D on l_2 is such that $AB = BD$.

Find the position vector of D. [5]

Sol (a) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 8i+5j-3k-5i-2j$

$= 3i+3j-3k$

$l_1 : r = 5i+2j+\lambda(i+j-k)$

(b) By comparing x, y and z - components of line l_1 and l_2

$5+\lambda = -2+3\mu$

$\lambda - 3\mu = -7$ — (i)

$2+\lambda = 1+\mu$

$\lambda - \mu = -1$ — (ii)

By solving eq(i) & eq(ii)

$\mu = 3 \quad \lambda = 2$

$l_1 \; r = 5i+2j+2(i+j-k)$

$= 7i+4j-2k$ ✓

$l_2 \; r = -2i+j+4k+3(3i+j-2k)$

$= 7i+4j-2k$ ✓

(c) let D point on l_2 $(-2+3\mu, 1+\mu, 4-2\mu)$ ✓

$$AB = OB - OA$$
$$= 5i + 2j - (-2i + j + 4k)$$
$$= 7i + J - 4K$$
$$|AB| = \sqrt{7^2 + 1^2 + (-4)^2} = \sqrt{66}$$

$$|AB| = |BD|$$
$$\therefore \quad BD = OD - OB = (3\mu - 7)i + (\mu - 1)j + (-2\mu + 4)k$$
$$|BD| = \sqrt{(3\mu - 7)^2 + (\mu - 1)^2 + (-2\mu + 4)^2} = \sqrt{66}$$
$$(3\mu - 7)^2 + (\mu - 1)^2 + (-2\mu + 4)^2 = 66$$
$$9\mu^2 - 42\mu + 49 + \mu^2 - 2\mu + 1 + 4\mu^2 - 16\mu + 16 = 66$$
$$14\mu^2 - 60\mu = 0 \qquad \underline{\underline{\mu = 0}} \qquad \mu = \frac{60}{14} = \frac{30}{7}$$

$$-2 + 3 \times \frac{30}{7} = \frac{76}{7}$$
$$1 + \frac{30}{7} = \frac{37}{7}$$
$$4 - 2 \times \frac{30}{7} = -\frac{32}{7}$$

Position Vector OD $\frac{76}{7}i + \frac{37}{7}J - \frac{32}{7}K$

Problem : 09709/32/M/J24/Q9

The complex numbers z and ω are defined by $z = 1 - \mathrm{i}$ and $\omega = -3 + 3\sqrt{3}\mathrm{i}$.

(a) Express $z\omega$ in the form $a + b\mathrm{i}$, where a and b are real and in exact surd form. [1]

(b) Express z and ω in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ in each case. [4]

(c) On an Argand diagram, the points representing ω and $z\omega$ are A and B respectively.

Prove that OAB is an isosceles right-angled triangle, where O is the origin. [2]

(d) Using your answers to part **(b)**, prove that $\tan \frac{5}{12}\pi = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. [3]

Sol (a) $zw = (1-i)(-3+3\sqrt{3}i)$

$= -3+3\sqrt{3}i + 3i - 3\sqrt{3}i^2 \qquad i^2 = -1$

$= -3 + 3\sqrt{3}i + 3i + 3\sqrt{3}$

$= (-3+3\sqrt{3}) + (3+3\sqrt{3})i$

(b) $z = 1-i$

$|z| = r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$

$w = -3+3\sqrt{3}i$

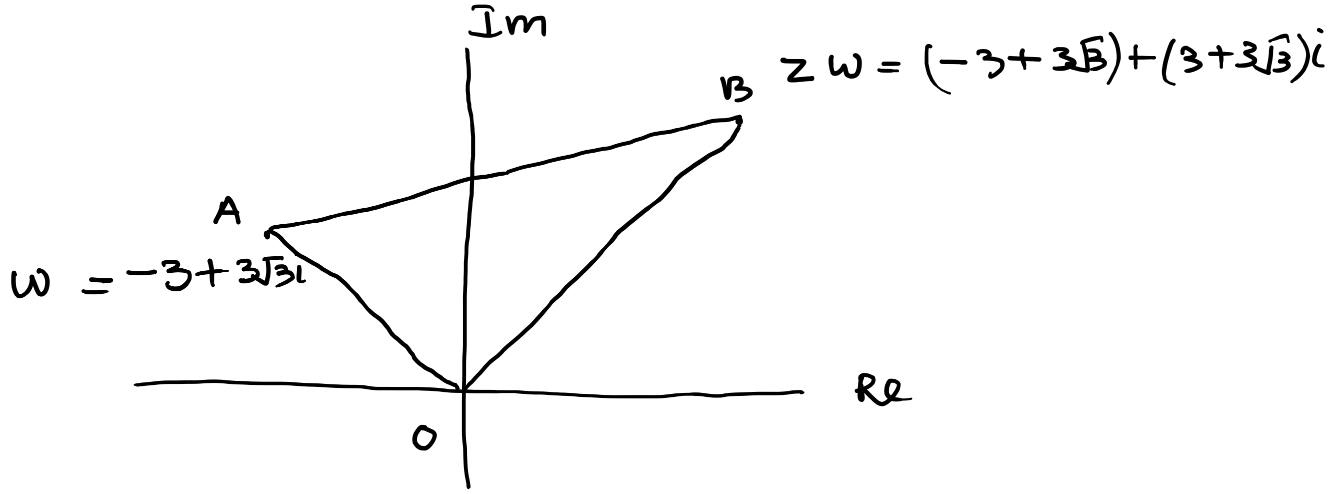
$|w| = \sqrt{(-3)^2+(3\sqrt{3})^2}$

$= \sqrt{9+27} = 6$

$\theta = \tan^{-1} \frac{3\sqrt{3}}{3} = \frac{\pi}{3}$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(c)



$|OA| = 6$

$$AB = -3+3\sqrt{3}+(3+3\sqrt{3})i-(-3+3\sqrt{3}i)$$
$$= -\cancel{3}+3\sqrt{3}+3i+\cancel{3\sqrt{3}i}+\cancel{3}-\cancel{3\sqrt{3}i}$$
$$= 3\sqrt{3}+3i$$

$$|AB| = \sqrt{(3\sqrt{3})^2+(3)^2}$$
$$= \sqrt{27+9}$$
$$= 6$$

$|OA| = |AB| = 6$

hence triangle is isosceles

$$\angle AOB = \text{Arg}\,w - \text{Arg}\,zw$$
$$= \cancel{\text{Arg}\,w} - \text{Arg}\,z - \cancel{\text{Arg}\,w}$$
$$= \frac{\pi}{4}$$

hence third angle is right angle.

(d) $$\text{Arg}\,zw = \text{Arg}\,z + \text{Arg}\,w$$
$$= \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$$

$$\text{Arg}\,zw = \tan^{-1}\frac{3+3\sqrt{3}}{-3+3\sqrt{3}}$$

$$\frac{5\pi}{12} = \tan^{-1} \frac{1+\sqrt{3}}{-1+\sqrt{3}}$$

$$\tan \frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Problem : 09709/32/M/J24/Q10

(a) By writing $y = \sec^3\theta$ as $\dfrac{1}{\cos^3\theta}$, show that $\dfrac{dy}{d\theta} = 3\sin\theta\sec^4\theta$. [2]

(b) The variables x and θ satisfy the differential equation

$$(x^2+9)\sin\theta\frac{d\theta}{dx} = (x+3)\cos^4\theta.$$

It is given that $x = 3$ when $\theta = \frac{1}{3}\pi$.

Solve the differential equation to find the value of $\cos\theta$ when $x = 0$. Give your answer correct to 3 significant figures. [8]

Sol (a) $y = \dfrac{1}{\cos^3\theta}$

$$\frac{dy}{d\theta} = \frac{0\times\cos^3\theta - 3\cos^2\theta.-\sin\theta}{(\cos^3\theta)^2}$$

$$= \frac{3\sin\theta\cos^2\theta}{\cos^6\theta}$$

$$= \frac{3\sin\theta}{\cos^4\theta} = 3\sin\theta.\sec^4\theta$$

(b) $$(x^2+9)\sin\theta\frac{d\theta}{dx} = (x+3)\cos^4\theta$$

By Variable seperable method

$$\int\frac{\sin\theta\, d\theta}{\cos^4\theta} = \int\frac{x+3}{x^2+9}\,dx$$

$$\int\sin\theta\sec^4\theta\, d\theta = \int\frac{x}{x^2+9}dx + \int\frac{3}{9+x^2}dx$$

by part (a)

$$\frac{1}{3}\sec^3\theta = \int\frac{x}{x^2+9}dx + \int\frac{3}{9+x^2}dx$$

$$\sec^3\theta = 3\int \frac{x}{x^2+9}\,dx + 9\int \frac{1}{9+x^2}\,dx$$

$$\sec^3\theta = 3 \cdot \frac{1}{2}\ln(x^2+9) + \frac{9}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$$

$u = x^2+9$

$du = 2x\,dx$

$$\int \frac{1\,dx}{1+\left(\frac{x}{3}\right)^2} = \tan^{-1}\frac{x}{3} \times 3$$

$$\int \frac{1}{u}\frac{du}{2}$$

$\frac{1}{2}\ln u$

$\frac{1}{2}\ln(x^2+9)$

$$\sec^3\theta = \frac{3}{2}\ln(x^2+9) + 3\tan^{-1}\frac{x}{3} + c$$

$x = 3 \quad \theta = \frac{1}{3}\pi$

$$\sec^3\left(\frac{\pi}{3}\right) = \frac{3}{2}\ln(3^2+9) + 3\tan^{-1}\frac{3}{3} + c$$

$$\frac{1}{\cos^3\left(\frac{\pi}{3}\right)} = \frac{3}{2}\ln 18 + \frac{3}{4}\pi + c$$

$$8 - \frac{3}{2}\ln 18 - \frac{3}{4}\pi = c$$

$c = 1.3082$

$$\sec^3\theta = \frac{3}{2}\ln(x^2+9) + 3\tan^{-1}\frac{x}{3} + 1.3082$$

$$\cos\theta = \sqrt[3]{\frac{1}{\frac{3}{2}\ln(x^2+9) + 3\tan^{-1}\frac{x}{3} + 1.3082}}$$

$x = 0$

$$\cos\theta = \frac{1}{\sqrt[3]{\frac{3}{2}\ln(9) + 0 + 1.3082}}$$

$= 0.601$