Problem : 09709/33/M/J24/Q1

Solve the equation $8^{3-6x} = 4 \times 5^{-2x}$. Give your answer correct to 3 decimal places. [4]

Sol

$$\frac{8^{3-6x}}{4} = 5^{-2x}$$

$$\frac{2^{3(3-6x)}}{2^2} = 5^{-2x}$$

$$2^{3(3-6x)-2} = 5^{-2x}$$

$$2^{9-18x-2} = 5^{-2x}$$

$$2^{7-18x} = 5^{-2x}$$

$$(7-18x)\ln 2 = -2x \ln 5$$

$$7\ln 2 - 18x\ln 2 = -2x\ln 5$$

$$7\ln 2 = 18x\ln 2 - 2x\ln 5$$

$$7\ln 2 = x\left[18\ln 2 - 2\ln 5\right]$$

$$x = \frac{7\ln 2}{18\ln 2 - 2\ln 5} = 0.524$$

Problem : 09709/33/M/J24/Q2

Find the exact coordinates of the stationary point of the curve $y = e^{2x}\sin 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. [5]

Sol $\frac{dy}{dx} = e^{2x}\cos 2x\ (2) + e^{2x}\cdot 2\sin 2x$

$0 = 2e^{2x}(\cos 2x + \sin 2x)$

$\cos 2x + \sin 2x = 0$ $\quad$ $e^{2x} = 0$ ✗

$\tan 2x = -1$

consider $\tan 2x = 1$

$2x = \tan^{-1} 1 = \frac{\pi}{4}$

$2x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

✓ $x = \frac{3\pi}{8}$

$y = e^{2\times\frac{3\pi}{8}} \sin 2\times\frac{3\pi}{8}$

$= e^{\frac{3\pi}{4}} \sin \frac{3\pi}{4}$

$= e^{\frac{3\pi}{4}} \frac{1}{\sqrt{2}}$

$= \frac{1}{2}\sqrt{2}e^{\frac{3}{4}\pi}$

$\left(\frac{3\pi}{8},\ \frac{1}{2}\sqrt{2}\,e^{\frac{3}{4}\pi}\right)$

Problem : 09709/33/M/J24/Q3

The square roots of $24-7\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $24-7\mathrm{i}$ in exact Cartesian form. [5]

Sol Let $\sqrt{24-7i} = a+ib$

$24-7i = (a+ib)^2$

$24-7i = a^2-b^2+2abi$

$a^2-b^2=24 \qquad 2ab=-7$

$b = -\dfrac{7}{2a}$ ✓

$a^2-\left(-\dfrac{7}{2a}\right)^2=24$

$a^2-\dfrac{49}{4a^2}=24$

$4a^4-96a^2-49=0$

let $a^2=t$

$4t^2-96t-49=0$

$t=\dfrac{49}{2} \qquad t=-\dfrac{1}{2}$

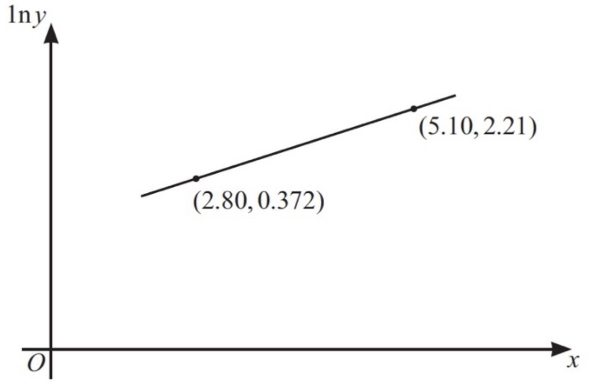
$a^2=\dfrac{49}{2} \qquad a^2=-1/2$ ✗

$a=\dfrac{7}{\sqrt{2}}$ or $\dfrac{7\sqrt{2}}{2} \qquad b=-\dfrac{7}{2\,\dfrac{7\sqrt{2}}{2}}=-\dfrac{\sqrt{2}}{2}$

$a=-\dfrac{7\sqrt{2}}{2} \qquad b=-\dfrac{7}{2\left(-\dfrac{7\sqrt{2}}{2}\right)}=\dfrac{\sqrt{2}}{2}$

$\dfrac{7\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$ and $-\dfrac{7\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$

Problem : 09709/33/M/J24/Q4



The variables x and y satisfy the equation $ky = e^{cx}$, where k and c are constants. The graph of $\ln y$ against x is a straight line passing through the points $(2.80, 0.372)$ and $(5.10, 2.21)$, as shown in the diagram.

Find the values of k and c. Give each value correct to 2 significant figures. [4]

Sol

$$ky = e^{cx}$$

$$\ln(ky) = cx \ln e$$

$$\ln k + \ln y = cx$$

$$\ln y = cx - \ln k$$

$$2.21 = c(5.10) - \ln k \quad \text{—(i)}$$

$$0.372 = c(2.80) - \ln k \quad \text{—(ii)}$$

By solving equations using elimination method.

$$1.838 = 2.3c$$

$$c = 0.7991$$

$$\approx 0.8$$

$$2.21 = 0.8 \times 5.1 - \ln k$$

$$\ln k = 0.8 \times 5.1 - 2.21$$

$$\ln k = 1.87$$

$$k = e^{1.87} = 6.488$$

$$\approx 6.5$$

Problem : 09709/33/M/J24/Q5

Express $\dfrac{6x^2-2x+2}{(x-1)(2x+1)}$ in partial fractions. [5]

Sol $\dfrac{6x^2-2x+2}{2x^2-x-1}$

$$2x^2-x-1 \,\big)\, 6x^2-2x+2 \,\big(\, 3$$
$$\underline{\cancel{6x^2}-3x-3}$$
$$x+5$$

$$3+\frac{x+5}{(x-1)(2x+1)}$$

$$\frac{x+5}{(x-1)(2x+1)} = \frac{A}{x-1}+\frac{B}{2x+1}$$

$$x+5 = A(2x+1)+B(x-1)$$

Substitute $x=1$

$$6 = A(3)$$

$$A = 2$$

Substitute $x=-1/2$

$$-\frac{1}{2}+5 = A\left(2\times -\frac{1}{2}+1\right)+B\left(-\frac{1}{2}-1\right)$$

$$\frac{9}{2} = 0+B\left(-\frac{3}{2}\right)$$

$$B = -3$$

$$3+\frac{2}{x-1}-\frac{3}{2x+1}$$

Problem : 09709/33/M/J24/Q6

(a) On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-4-3i| \leqslant 2$ and $\arg(z-2-i) \geqslant \frac{1}{3}\pi$. [5]

(b) Calculate the greatest value of $\arg z$ for points in this region. [2]

Sol (a) $(x-4)^2 + (y-3)^2 \leq 2$

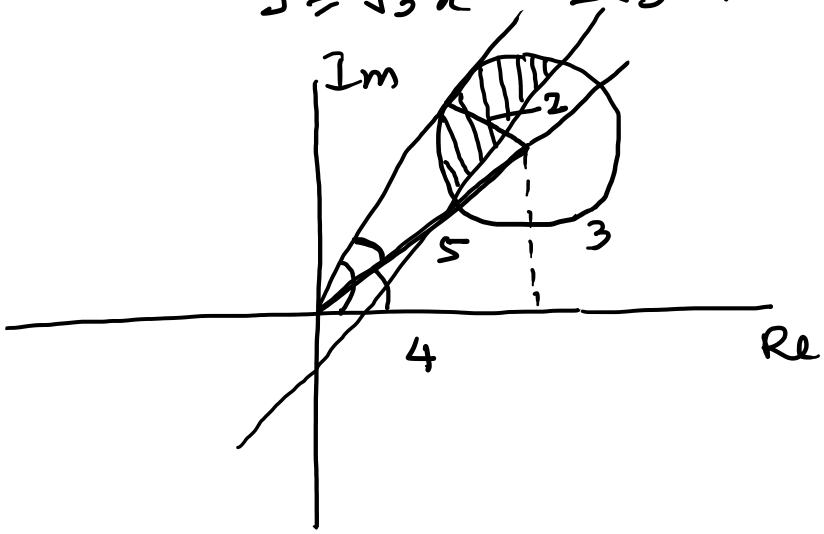
which is circle with centre $(4,3)$ and radius 2

$$\tan^{-1} \frac{y-1}{x-2} \geqslant \frac{1}{3}\pi$$

$$\frac{y-1}{x-2} \geqslant \tan \frac{\pi}{3}$$

$$y-1 \geqslant \sqrt{3}x - 2\sqrt{3}$$

$$y \geqslant \sqrt{3}x - 2\sqrt{3} + 1$$



(b) $\tan^{-1} \frac{3}{4} + \sin^{-1} \frac{2}{5}$

1.05

Problem : 09709/33/M/J24/Q7

Let $f(x) = 8x^3 + 54x^2 - 17x - 21$.

(a) Show that $x + 7$ is a factor of $f(x)$. [1]

(b) Find the quotient when $f(x)$ is divided by $x + 7$. [2]

(c) Hence solve the equation

$$8\cos^3\theta + 54\cos^2\theta - 17\cos\theta - 21 = 0,$$

for $0° \leqslant \theta \leqslant 360°$. [3]

Sol (a) $x+7=0$

$x=-7$

$f(-7) = 8(-7)^3 + 54(-7)^2 - 17(-7) - 21$

$= 0$

hence $x+7$ is factor of $f(x)$.

(b)

$$
\begin{array}{r|l}
x+7 & 8x^3 + 54x^2 - 17x - 21 \quad (8x^2 - 2x - 3 \\
 & \underline{-8x^3 \mp 56x^2} \\
 & -2x^2 - 17x \\
 & \underline{\mp 2x^2 \mp 14x} \\
 & -3x - 21 \\
 & \underline{-3x - 21} \\
 & \times
\end{array}
$$

$8x^2 - 2x - 3$

(c) $(x+7)(8x^2 - 2x - 3) = 0$

$(x+7)(8x^2 - 6x + 4x - 3) = 0$

$(x+7)(2x(4x-3) + 1(4x-3)) = 0$

$(x+7)(4x-3)(2x+1) = 0$

$x = -7 \quad x = \frac{3}{4} \quad x = -1/2$

$$8\cos^3\theta + 54\cos^2\theta - 17\cos\theta - 21 = 0$$

let $\cos\theta = x$

$$8x^3 + 54x^2 - 17x - 21 = 0$$

$x = -7$ $\quad x = \frac{3}{4}$ $\quad x = -1/2$

$\underline{\cos\theta = -7}$ $\quad \cos\theta = \frac{3}{4}$ $\quad \cos\theta = -\frac{1}{2}$

Out of Range $\quad \theta = \cos^{-1}\frac{3}{4}$ $\quad \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60^\circ$

$\underline{\theta = 41.4}$

$\underline{\theta = 318.6}$ $\quad \underline{\theta = 120^\circ}$

$\underline{\theta = 240^\circ}$

Problem : 09709/33/M/J24/Q8

(a) Express $3\cos 2x - \sqrt{3}\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and α. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{3}{(3\cos 2x - \sqrt{3}\sin 2x)^2}\,dx$, simplifying your answer. [5]

Sol (a) $R = \sqrt{3^2 + (-\sqrt{3})^2} = \sqrt{9+3}$

$= \sqrt{12}$

$\alpha = \tan^{-1} \frac{\sqrt{3}}{3}$

$\alpha = \frac{1}{6}\pi$

(b) $\int_0^{\frac{1}{12}\pi} \frac{3}{\left[\sqrt{12}\left(\cos\left(2x+\frac{\pi}{6}\right)\right)\right]^2} dx$

$\int_0^{\frac{1}{12}\pi} \frac{3}{12\cos^2\left(2x+\frac{\pi}{6}\right)} dx$

$\frac{1}{4}\int_0^{\frac{1}{12}\pi} \sec^2\left(2x+\frac{\pi}{6}\right) dx$

$\frac{1}{4}\left[\tan\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{1}{12}\pi} \times \frac{1}{2}$

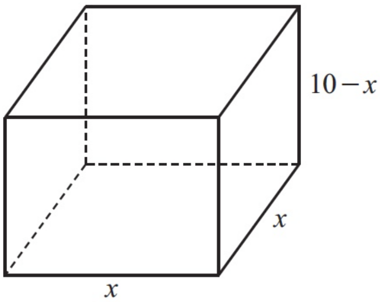
$\frac{1}{8}\left[\tan\left(\frac{\pi}{6}+\frac{\pi}{6}\right) - \tan\left(\frac{\pi}{6}\right)\right]$

$\frac{1}{8}\left[\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right)\right]$

$$\frac{1}{8}\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right] = \frac{1}{8}\left[\frac{2}{\sqrt{3}}\right]$$

$$= \frac{1}{4\sqrt{3}} = \frac{1}{\sqrt{48}}$$

Problem : 09709/33/M/J24/Q9



A container in the shape of a cuboid has a square base of side x and a height of $(10-x)$. It is given that x varies with time, t, where $t > 0$. The container decreases in volume at a rate which is inversely proportional to t.

When $t = \frac{1}{10}$, $x = \frac{1}{2}$ and the rate of decrease of x is $\frac{20}{37}$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{-1}{2t(20x - 3x^2)}.$$ [5]

(b) Solve the differential equation, obtaining an expression for t in terms of x. [6]

Sol (a) $V = x^2(10-x)$

$V = 10x^2 - x^3$

$$\frac{dV}{dt} = \frac{-k}{t}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dV}{dx} = 20x - 3x^2$$

$$20x - 3x^2 \times \frac{dx}{dt} = \frac{-k}{t}$$

$$\left(\overset{10}{\cancel{20}} \times \frac{1}{\cancel{2}} - 3 \times \frac{1}{4}\right) + \frac{20}{37} = \frac{+k}{1/10}$$

$$\left(10 - \frac{3}{4}\right)\frac{20}{37} = 10k$$

$$\frac{37}{4} \times \frac{20}{37} = 10k \qquad k = \frac{1}{2}$$

$$\frac{dv}{dt} = \frac{-1}{2t}$$

$$(20x-3x^2)\frac{dx}{dt} = \frac{-1}{2t}$$

$$\frac{dx}{dt} = \frac{-1}{2t\,(20x-3x^2)}$$

(b) Solve by variable seperable

$$\int (20x-3x^2)\,dx = -\frac{1}{2}\int \frac{1}{t}\,dt$$

$$20\frac{x^2}{2} - \frac{3x^3}{3} = -\frac{1}{2}\ln t + c$$

$t = \frac{1}{10}$ $x = \frac{1}{2}$

$$20\times\frac{1}{8} - \frac{1}{8} = -\frac{1}{2}\ln\frac{1}{10} + c$$

$$c = \frac{19}{8} + \frac{1}{2}\ln 0{\cdot}1$$

$$10x^2 - x^3 = -\frac{1}{2}\ln t + \frac{19}{8} + \frac{1}{2}\ln 0{\cdot}1$$

$$\frac{\ln t}{2} = x^3 - 10x^2 + \frac{19}{8} + \frac{1}{2}\ln 0{\cdot}1$$

$$\ln t = 2x^3 - 20x^2 + \frac{19}{4} + \ln 0{\cdot}1$$

$$t = e^{\left[2x^3 - 20x^2 + \frac{19}{4} + \ln 0{\cdot}1\right]}$$

$$t = \frac{1}{10}\,e^{2x^3 - 20x^2 + \frac{19}{4}}$$

Problem : 09709/33/M/J24/Q10

The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2a\mathbf{k} + \lambda(\underline{3\mathbf{i} + 4\mathbf{j} + a\mathbf{k}}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \mu(\underline{-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}),$$

where a is a constant.

(a) Given that the acute angle between the directions of these lines is $\frac{1}{4}\pi$, find the possible values of a. [6]

(b) Given instead that the lines intersect, find the value of a and the position vector of the point of intersection. [5]

Sol (a)

$$\cos \frac{1}{4}\pi = \frac{3 \times -1 + 4 \times 2 + a \times 2}{\sqrt{25+a^2} \times 3}$$

$$\frac{1}{\sqrt{2}} = \frac{-3+8+2a}{\sqrt{25+a^2} \times 3}$$

$$\frac{3}{\sqrt{2}} = \frac{5+2a}{\sqrt{25+a^2}}$$

$$\frac{9}{2} = \frac{25+4a^2+20a}{25+a^2}$$

$225 + 9a^2 = 50 + 8a^2 + 40a$

$a^2 - 40a + 175 = 0$

$a = 5 \quad a = 35$

(b) $1 + 3\lambda = -3 - \mu$

$3\lambda + \mu = -4$ — (i)

$1 + 4\lambda = -1 + 2\mu$

$4\lambda - 2\mu = -2$ — ii

by solving eq (i) & (ii)

$\lambda = -1 \quad \mu = -1$

$2a + a\lambda = 4 + 2\mu$

$2a - a = 2$

$\underline{\underline{a = 2}}$

$1 + 3(-1) = -2 \qquad -2$

$1 + 4(-1) = -3 \qquad -3$

$4 + 2(-1) = 2 \qquad 2$

$-3 + 1 = -2$

$-1 - 2 = -3$

$4 - 2 = 2$

hence position vector will be

$-2i - 3j + 2k$

Problem : 09709/33/M/J24/Q11

Use the substitution $2x = \tan\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{12}{\left(1+4x^2\right)^2}\,dx .$$

Give your answer in the form $a + b\pi$, where a and b are rational numbers. [9]

Sol $x = \frac{\tan\theta}{2}$

$dx = \frac{\sec^2\theta\, d\theta}{2}$

$x = 0 \quad 0 = \frac{\tan\theta}{2} \quad \theta = 0$

$x = 1/2 \quad \frac{1}{2} = \frac{\tan\theta}{2} \quad 1 = \tan\theta \quad \theta = \pi/4$

$$\overset{6}{\cancel{12}}\int_0^{\pi/4} \frac{1}{\left(1+\cancel{4}\frac{\tan^2\theta}{\cancel{4}}\right)^2}\frac{\sec^2\theta\, d\theta}{\cancel{2}}$$

$$6\int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta \qquad 6\int_0^{\pi/4} \frac{1}{(\sec^2\theta)^{\cancel{2}}}\cancel{\sec^2\theta}\, d\theta$$

$$6\int_0^{\pi/4} \cos^2\theta\, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$\frac{6}{2}\int_0^{\pi/4} (\cos 2\theta + 1)\, d\theta$$

$$3\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/4}$$

$$3\left[\frac{\sin \cancel{2}\times\frac{\pi}{\cancel{4}_2}}{2} + \frac{\pi}{4} - 0\right]$$

$$3\left[\frac{1}{2} + \frac{\pi}{4}\right]$$