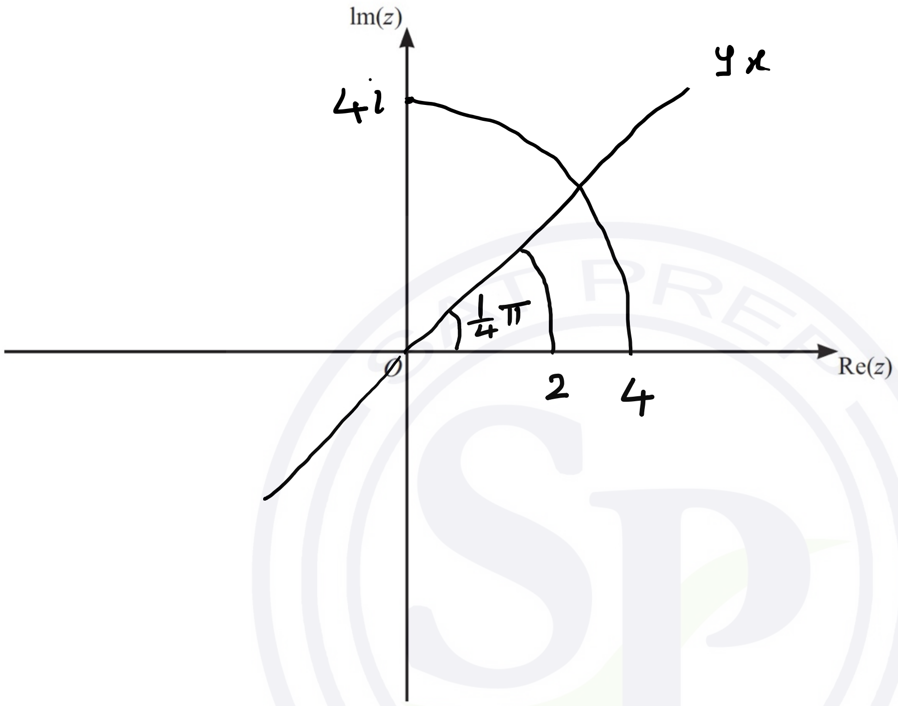
Problem : 09709/33/O/N24/Q1

The complex number z satisfies $|z| = 2$ and $0 \leqslant \arg z \leqslant \frac{1}{4}\pi$.

(a) On the Argand diagram below, sketch the locus of the points representing z. [2]

(b) On the **same diagram**, sketch the locus of the points representing z^2. [2]

$x^2 + y^2 = 2^2 \quad y = x$



Problem : 09709/33/O/N24/Q2

Let $f(x) = 2x^3 - 5x^2 + 4$.

(a) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\frac{4}{5-2x_n}}$$

converges, then it converges to a root of the equation $f(x) = 0$. [2]

(b) The equation has a root close to 1.2 .

Use the iterative formula from part **(a)** and an initial value of 1.2 to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a)

$$x = \sqrt{\frac{4}{5-2x}}$$

$$x^2 = \frac{4}{5-2x}$$

$$5x^2 - 2x^3 - 4 = 0$$

$$2x^3 - 5x^2 + 4 = 0$$

(b)

$$x_{n+1} = \sqrt{\frac{4}{5-2x_n}}$$

$x = 1.2$

$$= \sqrt{\frac{4}{5-2\times 1.2}} = 1.2403$$

$$= \sqrt{\frac{4}{5-2\times 1.2403}} = 1.2600$$

$= 1.2700$

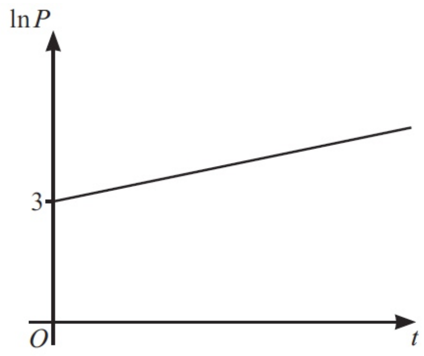
$= 1.2751$

$= 1.2778$

$= 1.2792$

Root of equation would be 1.28

Problem : 09709/33/O/N24/Q3



The number of bacteria in a population, P, at time t hours is modelled by the equation $P = ae^{kt}$, where a and k are constants. The graph of $\ln P$ against t, shown in the diagram, has gradient $\frac{1}{20}$ and intersects the vertical axis at $(0, 3)$.

(a) State the value of k and find the value of a correct to 2 significant figures. [3]

(b) Find the time taken for P to double. Give your answer correct to the nearest hour. [2]

Sol (a)

$$P = ae^{kt}$$

$$\ln P = \ln a + kt$$

0, 3 $\quad 3 = \ln a + 0$

$$a = e^3 = 20$$

$$k = \frac{1}{20}$$

(b)

$$40 = 20\, e^{1/20 t}$$

$$2 = e^{t/20}$$

$$\ln 2 = \frac{t}{20}$$

$$t = 20 \ln 2 = 13.86$$

$t = 14$ hours.

Problem : 09709/33/O/N24/Q4

Find the complex number z satisfying the equation

$$\frac{z-3i}{z+3i}=\frac{2-9i}{5}.$$

Give your answer in the form $x+iy$, where x and y are real. [5]

Sol $z = x+iy$

$$\frac{x+iy-3i}{x+iy+3i} = \frac{2-9i}{5}$$

$5x+5iy-15i = 2x-9ix+2iy+9y+6i+27$

by comparing real and imaginary part

Real $5x = 2x+9y+27$

$3x-9y = 27$ —(i)

Imaginary

$5y-15 = -9x+2y+6$

$9x+3y = 21$ —(ii)

Solving eq (i) & (ii) Simulteneously

$x=3$ $y=-2$

therefore $z = 3-2i$

Problem : 09709/33/O/N24/Q5

(a) Show that $\cos^4\theta - \sin^4\theta - 4\sin^2\theta\cos^2\theta \equiv \cos^2 2\theta + \cos 2\theta - 1$. [3]

(b) Solve the equation $\cos^4\alpha - \sin^4\alpha = 4\sin^2\alpha\cos^2\alpha$ for $0° \leqslant \alpha \leqslant 180°$. [3]

Sol (a) $\underbrace{(\cos^2\theta)^2 - (\sin^2\theta)^2}_{} - (2\sin\theta\cos\theta)^2$

$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) - (\sin 2\theta)^2$

$(1)(\cos 2\theta) - [1 - \cos^2 2\theta]$

$\cos 2\theta - 1 + \cos^2 2\theta$

$\cos^2 2\theta + \cos 2\theta - 1$ (RHS)

(b) $\cos^2 2\alpha + \cos 2\alpha - 1 = 0$

let $\cos 2\alpha = t$

$t^2 + t - 1 = 0$

$t = 0.618 \quad t = -1.618$ ×

$\cos 2\alpha = 0.618$

$2\alpha = \cos^{-1}(0.618)$

$2\alpha = 51.8$

$\alpha = 25.9°$

$2\alpha = 360 - 51.8$

$= 308.2$

$\alpha = 154.1°$

Problem : 09709/33/O/N24/Q6

The lines l and m have vector equations

$$l: \quad \mathbf{r} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{k}) \quad \text{and} \quad m: \quad \mathbf{r} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}+\mu(2\mathbf{i}-\mathbf{j}+5\mathbf{k}).$$

Lines l and m intersect at the point P.

(a) State the coordinates of P. [1]

(b) Find the exact value of the cosine of the acute angle between l and m. [3]

(c) The point A on line l has coordinates $(0,1,1)$. The point B on line m has coordinates $(0,2,-8)$.

Find the exact area of triangle APB. [3]

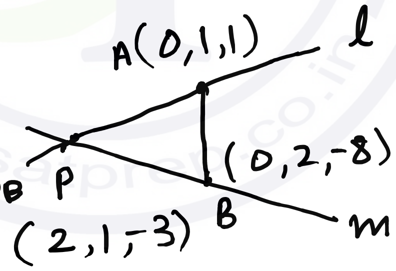
Sol (a) $P(2,1,-3)$

(b) $$\cos\theta = \frac{(-i+2k)\cdot(2i-j+5k)}{|(-i+2k)|\,.\,|(2i-j+5k)|}$$

$$= \frac{-2+10}{\sqrt{5}\,\sqrt{30}} = \frac{8}{\sqrt{5}\sqrt{30}}$$

$$= \frac{8}{5\sqrt{6}}$$

(c)



Area of $\triangle APB$

$= \frac{1}{2}|PA||PB|\sin APB$

$$PA = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\-3\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$|PA| = \sqrt{20}$

$$PB = \begin{pmatrix}0\\2\\-8\end{pmatrix} - \begin{pmatrix}2\\1\\-3\end{pmatrix} = \begin{pmatrix}-2\\1\\-5\end{pmatrix}$$

$|PB| = \sqrt{30}$

$$\sin APB = \sqrt{1-\left(\frac{8}{5\sqrt{6}}\right)^2}$$

$$= \sqrt{1-\frac{64}{150}}$$

$$= \sqrt{\frac{150-64}{150}}$$

$$= \frac{\sqrt{129}}{15}$$

$$\text{Area} = \frac{1}{2}\sqrt{20}\sqrt{30}\,\frac{\sqrt{129}}{15}$$

$$= \sqrt{86}$$

$= \frac{-2}{3}$ gradient of tangent

gradient of Normal $= \frac{3}{2}$

Eq of Normal

$$y - 2 = \frac{3}{2}(x - 3)$$

$$2y - 4 = 3x - 9$$

$$3x - 2y - 5 = 0$$

$$2y - 3x + 5 = 0$$

Problem : 09709/33/O/N24/Q8

Let $f(x) = \dfrac{7a^2}{(a-2x)(3a+x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [4]

(c) State the set of values of x for which the expansion in part **(b)** is valid. [1]

Sol (a) $\dfrac{7a^2}{(a-2x)(3a+x)} = \dfrac{A}{a-2x} + \dfrac{B}{3a+x}$

$7a^2 = A(3a+x) + B(a-2x)$

Substitute $x = -3a$

$7a^2 = A(0) + B(a-(-6a))$

$7a^2 = B(7a)$

$B = a$

Substitute $x = \dfrac{a}{2}$

$7a^2 = A\left(3a + \dfrac{a}{2}\right) + B(0)$

$7a^2 = \dfrac{7a}{2} A$

$A = 2a$

$\dfrac{2a}{a-2x} + \dfrac{a}{3a+x}$

(b) $2a(a-2x)^{-1} + a(3a+x)^{-1}$

$2\cancel{a} \cdot \cancel{a}^{-1}\left(1 - \dfrac{2x}{a}\right)^{-1} + a \cdot (3a)^{-1}\left(1 + \dfrac{x}{3a}\right)^{-1}$

$2\left[1 + (-1)\left(-\dfrac{2x}{a}\right) + \dfrac{-1(-1-1)}{2!}\left(-\dfrac{2x}{a}\right)^2\right]$

$+ \dfrac{1}{3}\left[1 + (-1)\left(\dfrac{x}{3a}\right) + \dfrac{-1(-1-1)}{2!}\left(\dfrac{x}{3a}\right)^2\right]$

$$2\left[1+\frac{2x}{a}+\frac{4x^2}{a^2}\right]+\frac{1}{3}\left[1-\frac{x}{3a}+\frac{x^2}{9a^2}\right]$$

$$2+\frac{4x}{a}+\frac{8x^2}{a^2}+\frac{1}{3}-\frac{x}{9a}+\frac{x^2}{27a^2}$$

$$\frac{7}{3}+\frac{35x}{9a}+\frac{217x^2}{27a^2}$$

(c) $$\left|\frac{2x}{a}\right|<1$$

$$|x|<\frac{a}{2}$$

Problem : 09709/33/O/N24/Q9

(a) Find the quotient and remainder when x^4+16 is divided by x^2+4. [3]

(b) Hence show that $\int_2^{2\sqrt{3}} \frac{x^4+16}{x^2+4}\,dx = \frac{4}{3}(\pi+4)$. [5]

Sol (a)

$$\begin{array}{r|l} x^2+4 & x^4+16 \quad (x^2-4 \\ & \underline{-x^4 - 4x^2} \\ & -4x^2+16 \\ & \underline{+4x^2+16} \\ & 32 \end{array}$$

Quotient x^2-4

Remainder 32

(b)

$$\int_2^{2\sqrt{3}} \left(x^2-4+\frac{32}{x^2+4}\right)dx$$

$$\left[\frac{1}{3}x^3-4x\right]_2^{2\sqrt{3}} + \frac{32}{2}\left[\tan^{-1}\frac{x}{2}\right]_2^{2\sqrt{3}}$$

$$\frac{1}{3}(2\sqrt{3})^3-4(2\sqrt{3})-\frac{1}{3}(2)^3+4(2)$$

$$+16\left[\tan^{-1}\frac{\cancel{2}\sqrt{3}}{\cancel{2}}-\tan^{-1}\frac{2}{2}\right]$$

$$8\cancel{\sqrt{3}}-\cancel{8\sqrt{3}}-\frac{8}{3}+8+16\left[\frac{\pi}{3}-\frac{\pi}{4}\right]$$

$$\frac{16}{3}+\frac{\overset{4}{\cancel{16}}}{\cancel{12}\,3}\pi$$

$$\frac{4\pi}{3}+\frac{16}{3}$$

$$\frac{4}{3}(\pi+4)$$

Problem : 09709/33/O/N/24/Q10

A water tank is in the shape of a cuboid with base area $40000\,\text{cm}^2$. At time t minutes the depth of water in the tank is h cm. Water is pumped into the tank at a rate of $50000\,\text{cm}^3$ per minute. Water is leaking out of the tank through a hole in the bottom at a rate of $600h\,\text{cm}^3$ per minute.

(a) Show that $200\frac{dh}{dt} = 250 - 3h$. [3]

(b) It is given that when $t = 0$, $h = 50$.

Find the time taken for the depth of water in the tank to reach 80 cm. Give your answer correct to 2 significant figures. [5]

Sol

$$V = 40000h$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dh} = 40000$$

$$50000 - 600h = 40000\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{\overset{250}{\cancel{50000}} - \overset{3}{\cancel{600}}h}{\underset{200}{\cancel{40000}}}$$

$$= \frac{250 - 3h}{200}$$

$$200\frac{dh}{dt} = 250 - 3h$$

(b) $$\int \frac{dh}{250-3h} = \int \frac{1}{200}t$$

$$\frac{\ln(250-3h)}{-3} = \frac{1}{200}t + c$$

$$-\frac{\ln(250-3h)}{3} = \frac{1}{200}t + c$$

$t=0 \quad h=50$

$$-\frac{\ln(250-150)}{3} = c$$

$$-\frac{\ln(250-3h)}{3} = \frac{t}{200} - \frac{\ln 100}{3}$$

$$\frac{1}{3}\ln 100 - \frac{1}{3}\ln(250-3h) = \frac{t}{200}$$

$$\frac{200}{3}\ln\left(\frac{100}{250-3h}\right) = t$$

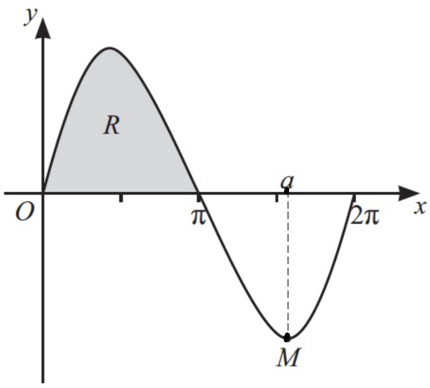
$h = 80$

$$t = \frac{200}{3}\ln\left(\frac{100}{250-3\times 80}\right)$$

$$= 153.5$$

$$= 150$$

Problem : 09709/33/O/N/24/Q11



The diagram shows the curve $y = 2\sin x\sqrt{2+\cos x}$, for $0 \leqslant x \leqslant 2\pi$, and its minimum point M, where $x = a$.

(a) Find the value of a correct to 2 decimal places. [5]

(b) Use the substitution $u = 2 + \cos x$ to find the exact area of the shaded region R. [6]

Sol (a) $\frac{dy}{dx} = 2\left[\sin x \cdot \frac{1}{2}(2+\cos x)^{-1/2} \times -\sin x + \cos x \cdot \sqrt{2+\cos x}\right]$

$$= \frac{-\sin^2 x}{\sqrt{2+\cos x}} + 2\cos x\sqrt{2+\cos x}$$

$$= \frac{-\sin^2 x + 2\cos(2+\cos x)}{\sqrt{2+\cos x}}$$

$$\frac{dy}{dx} = \frac{-\sin^2 x + 4\cos x + 2\cos^2 x}{\sqrt{2+\cos x}}$$

As M is stationary point $\therefore \frac{dy}{dx} = 0$

$$0 = -(1-\cos^2 x) + 4\cos x + 2\cos^2 x$$

$$0 = -1 + 4\cos x + 3\cos^2 x$$

let $\cos x = t$ $\quad 3t^2 + 4t - 1 = 0$

$t = 0.2152 \quad t = -1.5485$ ×

$\cos x = 0.2152 \quad x = 1.35$

$a = \underline{\underline{4.93}}$

(b) Area of $R = \int_0^{\pi} 2\sin x\sqrt{2+\cos x}\, dx$

$u = 2 + \cos x$

$du = -\sin x\, dx$

$x = 0 \quad u = 3$

$x = \pi \quad u = 1$

$A = -\int_3^1 2\, u^{1/2}\, du$

$= -\left[\frac{u^{3/2}}{3/2}\right]_3^1$

$= \frac{4}{3}\left[3^{3/2} - 1\right]$

$= \frac{4}{3}\left[3\sqrt{3} - 1\right]$