Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q1

The function f is defined by $f(x)=\dfrac{7x+7}{2x-4}$ for $x \in \mathbb{R}$, $x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y=f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Sol (a) $f(x)=0$

$$\frac{7x+7}{2x-4}=0$$

$$7x+7=0$$

$$7x=-7$$

$$x=-1$$

(b)

(i) $2x-4=0$

$$2x=4$$

$$x=2$$

(ii) $2x-4\,\big)\,7x+7\,\big(\,\frac{7}{2}$

$$\underline{\overset{-}{7x}\ \overset{+}{-}\ 14}$$

$$21$$

$$y=\frac{7}{2}$$

(c)

$$f(x)=\frac{7x+7}{2x-4}$$

Let $f(x)=y$

$$y=\frac{7x+7}{2x-4}$$

$$2xy-4y=7x+7$$

$$2xy-7x=4y+7$$

$$x(2y-7) = 4y+7$$

$$x = \frac{4y+7}{2y-7}$$

$$f^{-1}(x) = \frac{4x+7}{2x-7}$$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q2

On a Monday at an amusement park, a sample of 40 visitors was randomly selected as they were leaving the park. They were asked how many times that day they had been on a ride called *The Dragon*. This information is summarized in the following frequency table.

Number of times on *The Dragon*	**Frequency**
0	6
1	16
2	13
3	2
4	3

It can be assumed that this sample is representative of all visitors to the park for the following day.

(a) For the following day, Tuesday, estimate

(i) the probability that a randomly selected visitor will ride *The Dragon*;

(ii) the expected number of times a visitor will ride *The Dragon*. [4]

It is known that 1000 visitors will attend the amusement park on Tuesday. *The Dragon* can carry a maximum of 10 people each time it runs.

(b) Estimate the minimum number of times *The Dragon* must run to satisfy demand. [2]

Sol (a) (i) $P(\text{will ride The Dragon}) = 1 - \frac{6}{40}$

$= \frac{34}{40}$

(ii) $1 \times \frac{16}{40} + 2 \times \frac{13}{40} + 3 \times \frac{2}{40} + 4 \times \frac{3}{40}$

$= \frac{3}{2}$ or 1.5

(b) $1000 \times 1.5 = 150$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q3

Solve $\cos 2x = \sin x$, where $-\pi \le x \le \pi$.

Sol

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

let $\sin x = t$

$$2t^2 + t - 1 = 0$$

$$2t^2 + 2t - t - 1 = 0$$

$$2t(t+1) - 1(t+1) = 0$$

$$(2t-1)(t+1) = 0$$

$t = \frac{1}{2}$ $\quad$ $t = -1$

$\sin x = \frac{1}{2}$ $\quad$ $\sin x = -1$

$x = \sin^{-1}(-1)$

$= -\frac{\pi}{2}$

$x = \sin^{-1}\frac{1}{2}$

$x = \frac{\pi}{6}$

$x = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q4

Find the range of possible values of k such that $e^{2x} + \ln k = 3e^x$ has at least one real solution.

Sol

$(e^x)^2 - 3e^x + \ln K = 0$

Let $e^x = t$

$t^2 - 3t + \ln K = 0$

$b^2 - 4ac \geq 0$

$(-3)^2 - 4 \times 1 \times \ln K \geq 0$

$9 - 4 \ln K \geq 0$

$9 \geq 4 \ln K$

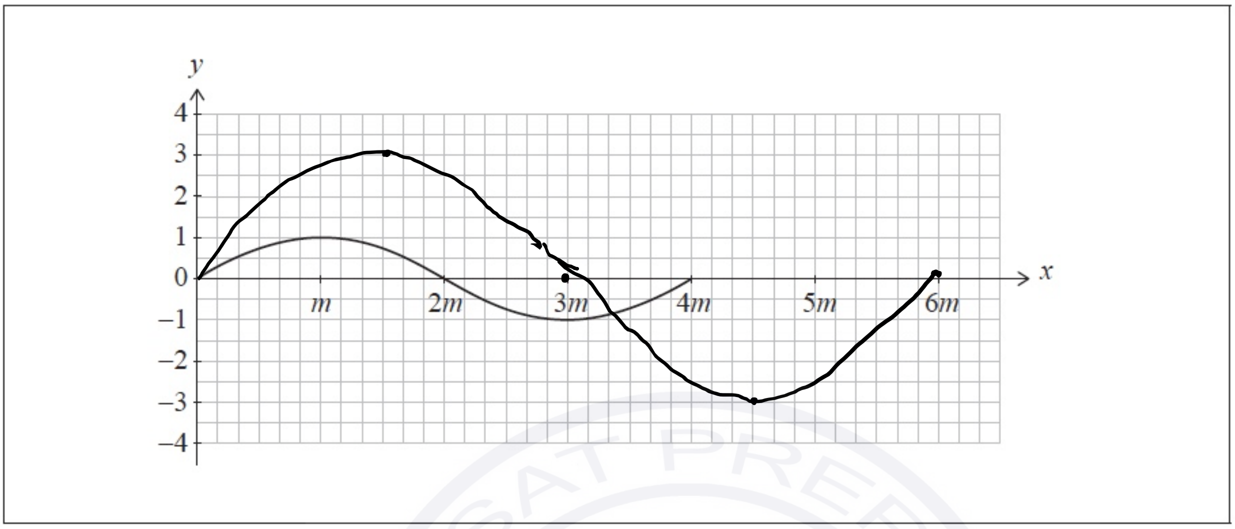
$\ln K \leq \frac{9}{4}$

$K \leq e^{9/4}$

$0 < K \leq e^{9/4}$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q5

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \le x \le 4m$, where x is in radians. There are x-intercepts at $x = 0$, $2m$ and $4m$.



(a) Find an expression for m in terms of q. [2]

The function g is defined by $g(x) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of g. [4]

Sol (a)

$$q = \frac{2\pi}{\text{period}}$$

$$q = \frac{2\pi}{4m}$$

$$q = \frac{\pi}{2m} \checkmark$$

$$m = \frac{\pi}{2q}$$

(b)

$$g(x) = \underline{3}\sin\frac{2q}{3}x \quad , 0 \le x \le 6m$$

Vertical stretch or amplitude = 3

period = 0 to 6 m

$$\sin 2\frac{\pi}{3\,2m} = \sin\frac{\pi}{3m}$$

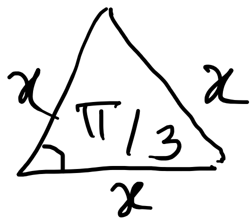
Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q6

The side lengths, $x\,\text{cm}$, of an equilateral triangle are increasing at a rate of $4\,\text{cm}\,\text{s}^{-1}$.

Find the rate at which the area of the triangle, $A\,\text{cm}^2$, is increasing when the side lengths are $5\sqrt{3}$ cm.

Sol

$$A = \frac{1}{2}x^2 \sin\frac{\pi}{3}$$



$$\frac{dA}{dx} = x\sin\frac{\pi}{3}$$

$$\frac{dA}{dt} = \frac{dA}{dx}\times\frac{dx}{dt}$$

$$\frac{dA}{dt} = x\sin\frac{\pi}{3}\times 4$$

$$x = 5\sqrt{3}$$

$$\frac{dA}{dt} = 5\sqrt{3}\times\frac{\sqrt{3}}{\cancel{2}}\times\cancel{4}^{2}$$

$$= 30\ \text{cm}^2$$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q7

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$.

Sol $\quad z = 3i \quad z = -3i$

$z = 3i$

$P(3i) = 4m - m(3i) + \frac{36}{m}(3i)^2 - (3i)^3$

$0 = 4m - 3mi - \frac{36}{m} \times 9 + 27i$

$0 = 4m - \frac{36}{m} \times 9 + (27 - 3m)i$

$27 - 3m = 0$

$m = \frac{27}{3} = 9$

$P(z) = 36 - 9z + \frac{36}{9}z^2 - z^3$

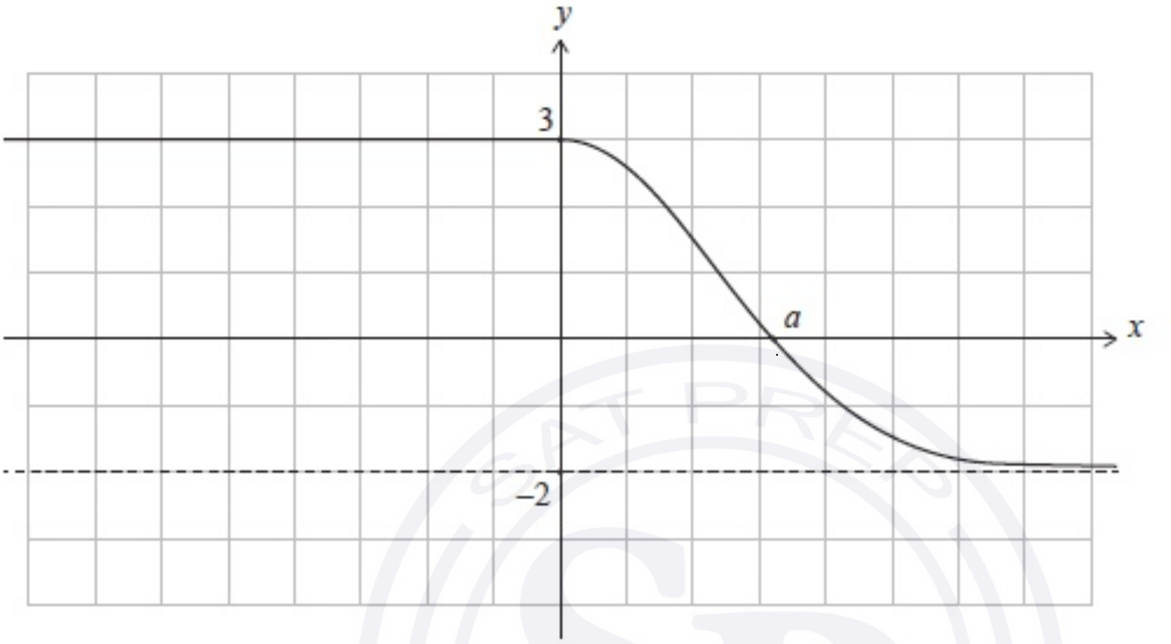
$= 36 - 9z + 4z^2 - z^3$

$0 = (z^2 + 9)\left(\frac{36}{9} - z\right)$

$z = -3i, 3i$ and 4

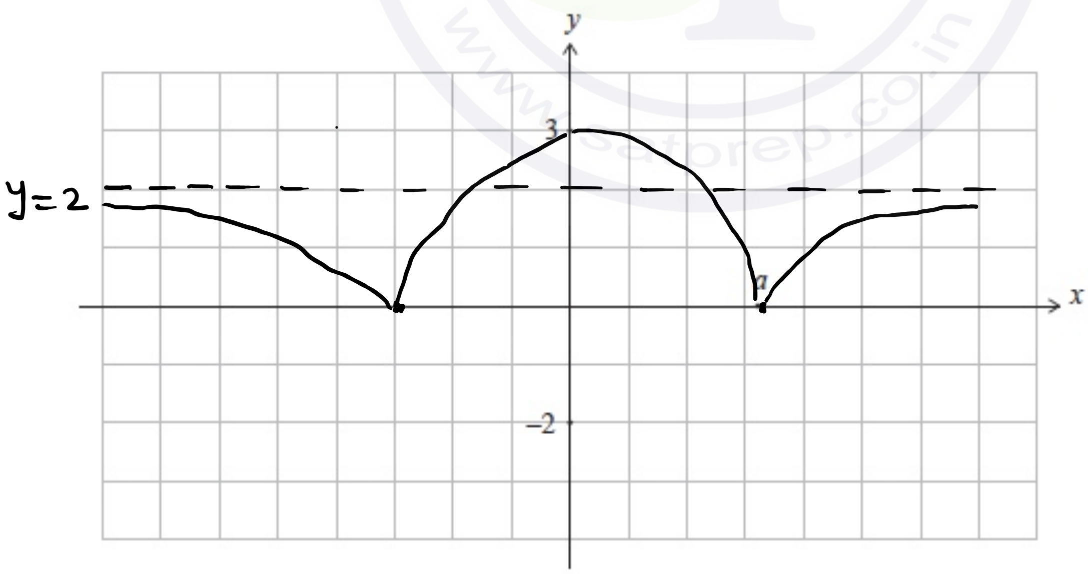
Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q8

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0, 3)$, an x-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.



Consider the function $g(x) = |f(|x|)|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]



(b) Find the possible values of k such that $(g(x))^2 = k$ has exactly two solutions. [3]

$k = 0$

$4 \leq k < 9$

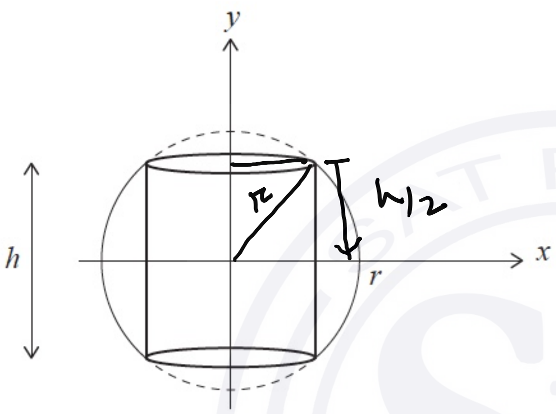
Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q9

The function f is defined by $f(y)=\sqrt{r^2-y^2}$ for $-r \le y \le r$.

The region enclosed by the graph of $x=f(y)$ and the y-axis is rotated by $360°$ about the y-axis to form a solid sphere. The sphere is drilled through along the y-axis, creating a cylindrical hole. The resulting spherical ring has height, h.

This information is shown in the following diagrams.

diagram not to scale



The spherical ring has a volume of π cubic units. Find the value of h.

Sol radius of cylinder $= \sqrt{r^2 - \frac{h^2}{4}}$

and $y=0$ to $y=h/2$

Now volume of sphere

$$= 2\pi \int_0^{h/2} (r^2 - y^2)\,dy$$

$$= 2\pi \left[r^2 y - y^3/3 \right]_0^{h/2}$$

$$= 2\pi \left[\frac{r^2 h}{2} - \frac{h^3}{24} \right]$$

Volume of Cylinder $= \pi\left(r^2 - \frac{h^2}{4}\right)h$

Volume of spherical ring

= Volume of sphere − Volume of Cylinder

$$\cancel{\pi} = 2\cancel{\pi}\left[\frac{r^2h}{2} - \frac{h^3}{24}\right] - \cancel{\pi}\left(r^2 - \frac{h^2}{4}\right)h$$

$$1 = \cancel{r^2h} - \frac{h^3}{12} - \cancel{r^2h} + \frac{h^3}{4}$$

$$1 = \frac{h^3}{4} - \frac{h^3}{12}$$

$$1 = \frac{3h^3 - h^3}{12}$$

$$12 = 2h^3$$

$$h^3 = 6$$

$$h = \sqrt[3]{6}$$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q10

Consider the arithmetic sequence $u_1, u_2, u_3, \dots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find u_6. [4]

(b) Find u_1. [2]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

(e) Given that $v_{99} < 0$, find v_5. [2]

Sol a(i) $S_5 = (5)^2 + 4(5) = 45$

(ii) $U_6 = S_6 - S_5$

$= 60 - 45 = 15$

(b) $U_1 = S_1 = (1)^2 + 4(1) = 5$

(c) $U_n = U_1 + (n-1)d$

$= 5 + (n-1)2$

$= 5 + 2n - 2$

$U_n = 2n + 3$

$U_2 = S_2 - S_1$

$= (2)^2 + 4(2) - 5$

$U_2 = 7$

$d = U_2 - U_1$

$= 7 - 5$

$= 2$

(d) $V_2 = U_1$

$a \cdot r = 5$ —(i)

$a \cdot r^3 = 15$ —(ii)

Divide eq(ii) by eq(i)

$\frac{ar^3}{ar} = \frac{15}{5}$

$a = \frac{5}{-\sqrt{3}}$

$r^2 = 3$

$r = \pm\sqrt{3}$

(e) $V_{99} < 0$

Consider $r = -\sqrt{3}$

$$U_5 = a \cdot r^4$$

$$= \frac{5}{-\sqrt{3}} \times \left(-\sqrt{3}\right)^4$$

$$= -\frac{5}{\sqrt{3}} \times 9$$

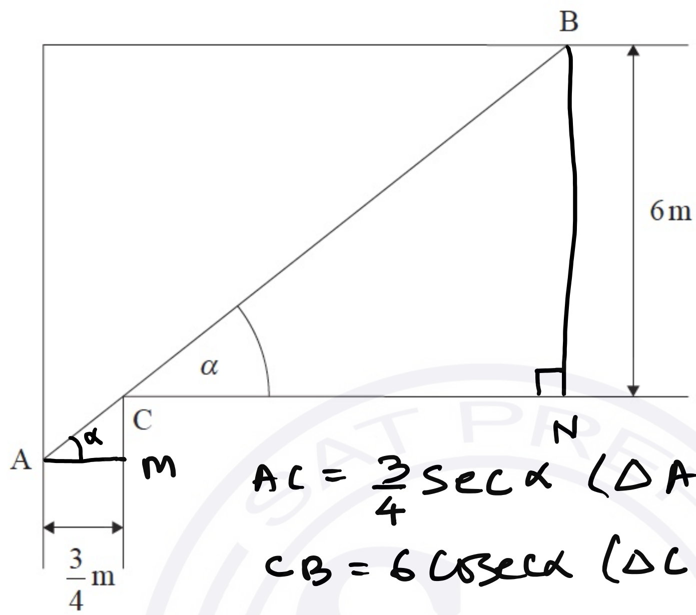
$$= -\frac{5 \times \cancel{9}^{\,3} \times \sqrt{3}}{\cancel{3}}$$

$$U_5 = -15\sqrt{3}$$

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q11

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



$AC = \frac{3}{4}\sec\alpha$ (ΔAMC)

$CB = 6\,\text{cosec}\,\alpha$ (ΔCNB)

A narrow passageway with width $\frac{3}{4}\,\text{m}$ is perpendicular to a room of width $6\,\text{m}$. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that $[\text{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\alpha + 6\,\text{cosec}\,\alpha$. [2]

(b) (i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$.

(ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan 2$. [5]

(c) (i) Find $\dfrac{d^2L}{d\alpha^2}$.

(ii) When $\alpha = \arctan 2$, show that $\dfrac{d^2L}{d\alpha^2} = \dfrac{45}{4}\sqrt{5}$. [7]

(d) (i) Hence, justify that L is a minimum when $\alpha = \arctan 2$.

(ii) Determine this minimum value of L. [3]

Two people need to carry a pole of length $11.25\,\mathrm{m}$ from the passageway into the room.
It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

Sol (a)

$$L = AB$$

$$= AC + CB$$

$$L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$$

(b) (i)

$$\frac{dL}{d\alpha} = \frac{3}{4}\sec\alpha.\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$$

(ii)

$$\frac{dL}{d\alpha} = 0$$

$$\overset{2}{\cancel{6}}\operatorname{cosec}\alpha.\cot\alpha = \frac{\cancel{3}}{4}\sec\alpha.\tan\alpha$$

$$2\,\frac{\cos\alpha}{\sin^2\alpha} = \frac{\sin\alpha}{4\cos^2\alpha}$$

$$8\cos^3\alpha = \sin^3\alpha$$

$$8 = \tan^3\alpha$$

$$\tan\alpha = 2$$

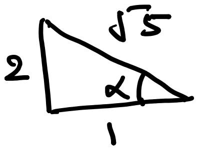
$$\alpha = \arctan 2$$

c(i) $$\frac{d^2L}{d\alpha^2} = \frac{3}{4}\left[\sec\alpha.\sec^2\alpha + \tan\alpha\sec\alpha.\tan\alpha\right] - 6\left[\operatorname{cosec}\alpha.(-\operatorname{cosec}^2\alpha) + \cot\alpha(-\operatorname{cosec}\alpha.\cot\alpha)\right]$$

$$= \frac{3}{4}\left[\sec^3\alpha + \tan^2\alpha\sec\alpha\right] + 6\operatorname{cosec}^3\alpha + 6\cot^2\alpha\operatorname{cosec}\alpha$$

$$\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec^3\alpha + \frac{3}{4}\tan^2\alpha\sec\alpha + 6\operatorname{cosec}^3\alpha + 6\cot^2\alpha.\operatorname{cosec}\alpha$$

c(ii) $\alpha = \text{Arctan}\,2$



$\tan\alpha = 2$

$\cot\alpha = \frac{1}{2}$

$\sec\alpha = \sqrt{5}$

$\operatorname{cosec}\alpha = \frac{\sqrt{5}}{2}$

$$\frac{d^2L}{d\alpha^2} = \frac{3}{4}(\sqrt{5})^3 + \frac{3}{4}(2)^2(\sqrt{5}) + 6\left(\frac{\sqrt{5}}{2}\right)^3 + 6\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{5}}{2}\right)$$

$$= \frac{3}{4}5\sqrt{5} + \frac{3}{4}\times 4\times\sqrt{5}) + 6\frac{5\sqrt{5}}{8} + \frac{6}{4}\frac{\sqrt{5}}{2}$$

$$= \frac{15\sqrt{5}}{4} + \frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4}$$

$$= \frac{45\sqrt{5}}{4}$$

(d)(i) Since $\frac{d^2L}{d\alpha^2} = \frac{45\sqrt{5}}{4} > 0$

hence L is minimum

at $\alpha = \text{Arctan}\,2$

(ii) $$L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$$

$$= \frac{3}{4}\sqrt{5} + 6\frac{\sqrt{5}}{2}$$

$$= \frac{15\sqrt{5}}{4}$$

(e) $L = 11.25$ $L_{min} = \frac{15\sqrt{5}}{4}$

$11.25 > \frac{15\sqrt{5}}{4}$

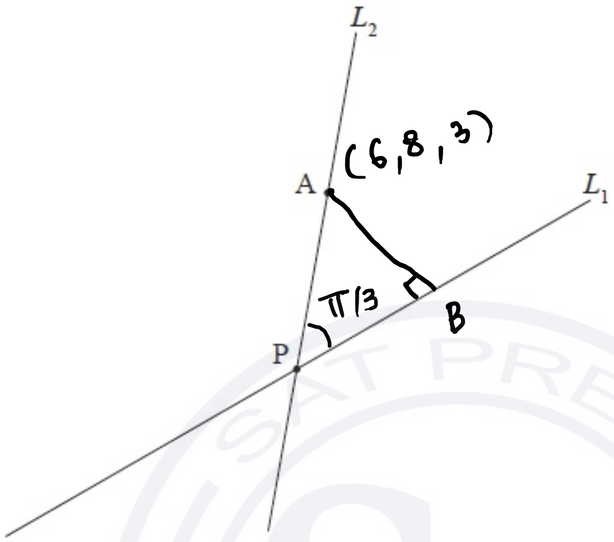
So pole cannot be carried

Problem - M23/5/MATHX/HP1/ENG/TZ1/XX/Q12

Two lines, L_1 and L_2, intersect at point P. Point $\text{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

diagram not to scale

As $t = 3$ (part b)



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\overrightarrow{\text{PA}} = \begin{pmatrix}2t\\0\\3+t\end{pmatrix}$. $\begin{pmatrix}6\\0\\6\end{pmatrix}$ by $t = 3$ (Part b)

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of t. [4]

(c) Hence or otherwise, find the shortest distance from A to L_1. [4]

A plane, Π, contains L_1 and L_2.

(d) Find a normal vector to Π. [2]

The base of a right cone lies in Π, centred at A such that L_1 is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

Sol (a) $\cos\frac{\pi}{3} = \frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2t\\0\\3+t\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\right\| \left\|\begin{pmatrix}2t\\0\\3+t\end{pmatrix}\right\|}$

$$\frac{1}{2} = \frac{2t}{\sqrt{2}\sqrt{4t^2+(3+t)^2}}$$

$$\frac{1}{2} = \frac{2t}{\sqrt{2}\sqrt{4t^2+9+6t+t^2}}$$

$$1 = \frac{4t}{\sqrt{2}\sqrt{5t^2+6t+9}}$$

$$4t = \sqrt{10t^2+12t+18}$$

(b) Square both sides of answer of part (a)

$$16t^2 = 10t^2+12t+18$$

$$6t^2-12t-18=0$$

$$t^2-2t-3=0$$

$$t^2-3t+t-3=0$$

$$t(t-3)+1(t-3)=0$$

$$(t-3)(t+1)=0$$

$\underline{\underline{t=3}}$ $\quad t=-1$

(c) In $\triangle PBA$

$$\sin\frac{\pi}{3} = \frac{AB}{|\vec{PA}|}$$

$$AB = |\vec{PA}| \times \sin\frac{\pi}{3}$$

$$AB = \left\| \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} \right\| \times \sin\frac{\pi}{3}$$

$$= \sqrt{36+0+36} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{72} \times \sqrt{3}}{2}$$

$$= \frac{\cancel{2}\sqrt{2} \times 3 \times \sqrt{3}}{\cancel{2}}$$

$$= 3\sqrt{6}$$

(d) Use cross product between L_1 and L_2

$$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 6 & 0 & 6 \end{vmatrix}$$

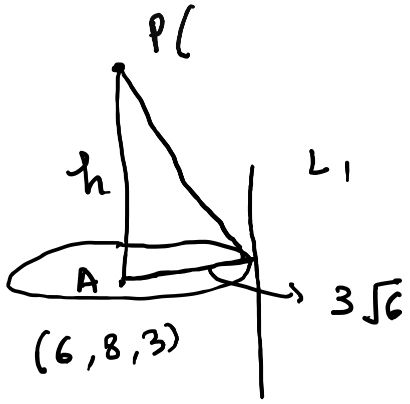
$$= i(6-0) + j(0-6) + k(0-6)$$

$$= i(6) - 6j - 6k$$

$$= i - j - k$$

$$= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(e)



$$90\pi\sqrt{3} = \frac{1}{3}\pi r^2 h$$

$$90\cancel{\pi}\sqrt{3} = \frac{1}{3}\cancel{\pi}(3\sqrt{6})^2 h$$

$$10\ \cancel{90}\sqrt{3} = \frac{\cancel{9} \times 6}{\cancel{3}}^{2} h$$

$$h = 5\sqrt{3}$$

$$\vec{PA} = \begin{pmatrix} 2t \\ 0 \\ 3+t \end{pmatrix}$$

$t = 3 \qquad \vec{PA} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$

therefore
vector equation PA

$$= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$h = \mu \, |\hat{n}|$$

$$5\sqrt{3} = \mu \sqrt{3}$$

square both sides

$$25 \times \cancel{3} = \mu^2 \times \cancel{3}$$

$$\mu = \pm 5$$

$\mu = 5$

$$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ -2 \end{pmatrix}$$

$\mu = -5$

$$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + (-5) \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \\ 8 \end{pmatrix}$$