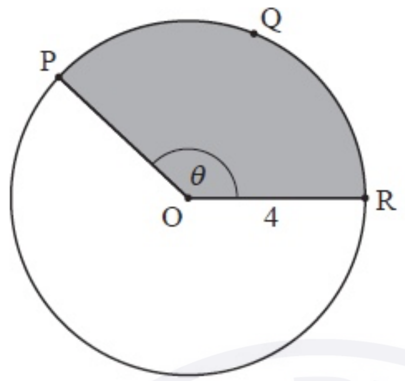
Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q1

The following diagram shows a circle with centre O and radius 4 cm.

diagram not to scale



The points P, Q and R lie on the circumference of the circle and $P\hat{O}R = \theta$, where θ is measured in radians.

The length of arc PQR is 10 cm.

(a) Find the perimeter of the shaded sector. [2]

(b) Find θ. [2]

(c) Find the area of the shaded sector. [2]

Sol (a) Perimeter of shaded sector

$= 2 \times \text{radius} + \text{Arc length of sector}$

$= 2 \times 4 + 10 = 18\text{cm}$

(b) Arc length of shaded sector

$= r\theta$

$10 = 4 \times \theta$

$\theta = \frac{10}{4} = \frac{5}{2}$ radians

(c) Area of shaded sector

$= \frac{1}{2} r^2 \theta$

$= \frac{1}{\cancel{2}} \times \cancelto{4}{16} \times \frac{5}{\cancel{2}} = 20\text{ cm}^2$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q2

A function f is defined by $f(x)=1-\dfrac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

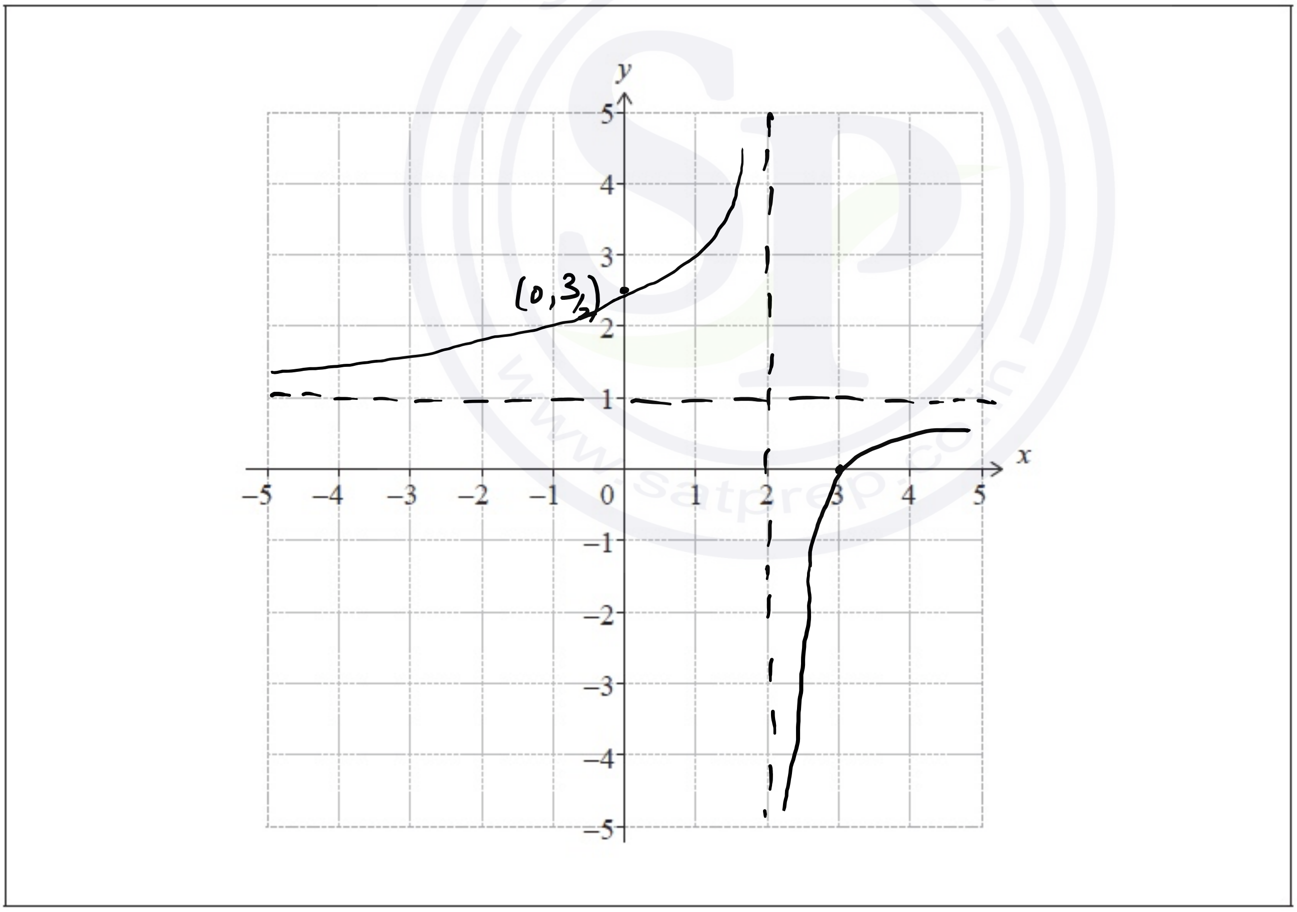
(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y=f(x)$ intersects

(i) the y-axis;

(ii) the x-axis. [2]

(c) On the following set of axes, sketch the graph of $y=f(x)$, showing all the features found in parts (a) and (b). [1]



Sol (a) (i) $x-2=0$

$x=2$

(ii) $y=1$

(b) (i) Y-axis intercept

$x = 0$ in $f(x)$

$$f(x) = 1 - \frac{1}{x-2}$$

$$f(0) = 1 - \frac{1}{0-2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$\left(0, \frac{3}{2}\right)$

(ii) X-axis intercept

$y = f(x) = 0$

$$0 = 1 - \frac{1}{x-2}$$

$$\frac{1}{x-2} = 1$$

$x - 2 = 1$

$x = 3$

$(3, 0)$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q3

Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A|B) = 0.25$ and $\mathrm{P}(A \cup B) = 0.55$.

Find $\mathrm{P}(B)$.

<u>Sol</u>

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B) \times P(A|B)$$

$$= P(B) \times 0.25$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

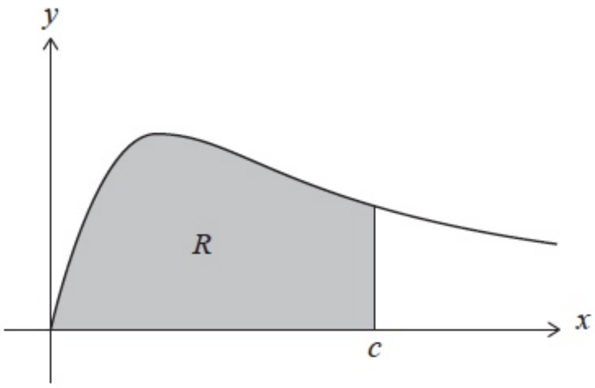
$$0.55 = 0.4 + P(B) - P(B) \times 0.25$$

$$0.55 = 0.4 + P(B)(0.75)$$

$$P(B) = \frac{0.15}{0.75} = \frac{1}{5} = 0.2$$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q4

The following diagram shows part of the graph of $y = \frac{x}{x^2+2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c.

Sol

$$\int_0^c \frac{x}{x^2+2}\,dx = \ln 3$$

let $x^2+2 = t$

$$2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$x = 0 \quad t = 2$

$x = c \quad t = c^2+2$

$$\frac{1}{2}\int_2^{c^2+2} \frac{1}{t}\,dt = \ln 3$$

$$\frac{1}{2}\left[\ln t\right]_2^{c^2+2} = \ln 3$$

$$\ln(c^2+2) - \ln 2 = 2\ln 3$$

$$\ln \frac{c^2+2}{2} = \ln 9$$

$$c^2+2 = 18$$

$$c^2 = 16$$

$c = \pm 4$

$c = 4$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q5

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b\text{, where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3.$$

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

Sol $(g \circ f)(x) = (ax+b)^2 + (ax+b) + 3$

$= a^2x^2 + b^2 + 2abx + ax + b + 3$

$4x^2 - 14x + 15 = a^2x^2 + (2ab + a)x + b^2 + b + 3$

$a^2 = 4 \quad a = \pm 2$

$b^2 + b + 3 = 15$

$b^2 + b - 12 = 0$

$b^2 + 4b - 3b - 12 = 0$

$b(b+4) - 3(b+4) = 0$

$(b+4)(b-3) = 0$

$b = -4 \quad b = 3$

$a = \pm 2 \quad b = -4, 3$

$f(x) = 2x - 4 \qquad f(x) = 2x + 3$

$f(x) = -2x - 4 \qquad f(x) = -2x + 3$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q6

A continuous random variable X has probability density function f defined by

$$f(x)=\begin{cases}\frac{1}{2a}, & a\le x\le 3a \\ 0, & \text{otherwise}\end{cases}$$

where a is a positive real number.

(a) State $\mathrm{E}(X)$ in terms of a. [1]

(b) Use integration to find $\mathrm{Var}(X)$ in terms of a. [4]

Sol (a) $E(x)=\frac{3a+a}{2}=2a$ (by symmetry)

(b) $Var(x)=E(x^2)-[E(x)]^2$

$$=\int_a^{3a}\frac{1}{2a}.x^2\,dx-(2a)^2$$

$$=\frac{1}{2a}\left[\frac{x^3}{3}\right]_a^{3a}-4a^2$$

$$=\frac{1}{6a}\left[27a^3-a^3\right]-4a^2$$

$$=\frac{1}{6a}\times 26a^3-4a^2$$

$$=\frac{26a^3-24a^2}{6a}$$

$$=\frac{2a^3}{6a}=\frac{a^2}{3}$$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q7

Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

Sol let $P(n)$ be the proposition that

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

Consider $P(1)$ $n=1$

LHS $\frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$

RHS $1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2}$

So $P(1)$ is true.

Assume $P(k)$ is true i.e

$$\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$$

Consider $P(k+1)$ so $n = k+1$

$$\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{((k+1)+1)!}$$

$$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+1)!} \times \frac{k+2}{k+2} + \frac{k+1}{(k+2)!}$$

$(n+1).\, n! = (n+1)!$

$$= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!}$$

$$= 1 - \left[\frac{\cancel{k}+2-\cancel{k}-1}{(k+2)!} \right]$$

$$= 1 - \frac{1}{(k+2)!}$$

$$= 1 - \frac{1}{((k+1)+1)!}$$

$P(k+1)$ is true whenever $P(k)$ is true and $P(1)$ is true. So $P(n)$ is true for all integers, $n \geq 1$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q8

The functions f and g are defined by

$$f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \le x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a+\sqrt{b}}{c}$, where $a,\ c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

Sol (a) $\cos x = \tan x$

$\cos k = \tan k$

$\cos k = \frac{\sin k}{\cos k}$

$\cos^2 k = \sin k$ ✓

(b) $m_1 = f'(x) = -\sin x$

$m_2 = g'(x) = \sec^2 x$

$m_1 \times m_2 = -1$

$-\sin x \times \sec^2 x$

$-\sin x \times \frac{1}{\cos^2 x}$

$-\sin k \times \frac{1}{\cos^2 k}$

by result of part(a)

$-\cancel{\sin k} \times \frac{1}{\cancel{\sin k}} = -1$

(c) By result of Part (a)

$$\cos^2 k = \sin k$$

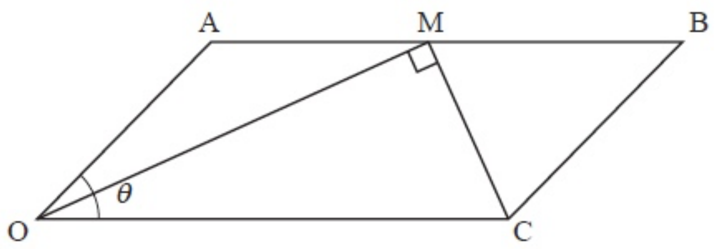
$$1 - \sin^2 k - \sin k = 0$$

$$\sin^2 k + \sin k - 1 = 0$$

$$\sin k = \frac{-1 + \sqrt{5}}{2}$$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q9

The following diagram shows parallelogram OABC with $\vec{OA} = \boldsymbol{a}$, $\vec{OC} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \neq 0$.



$\vec{OC} \parallel \vec{AB}$

$\therefore \vec{AB} = c$

$\vec{AM} = kc$

$\vec{OA} = a$

The angle between $\vec{OA}$ and $\vec{OC}$ is θ, where $0 < \theta < \pi$.

Point M is on [AB] such that $\vec{AM} = k\vec{AB}$, where $0 \le k \le 1$ and $\vec{OM} \cdot \vec{MC} = 0$.

(a) Express $\vec{OM}$ and $\vec{MC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$. [3]

(c) Find the range of values for θ such that there are two possible positions for M. [4]

Sol (a)

$$\vec{OM} = \vec{OA} + \vec{AM}$$
$$= a + kc$$
$$\vec{MC} = \vec{OC} - \vec{OM}$$
$$= c - a - kc$$
$$= (1-k)c - a$$

(b)

$$\vec{OM} \cdot \vec{MC} = 0$$
$$(a+kc)((1-k)c - a) = 0$$
$$(1-k)\,a.c - a.a + k(1-k)\,c.c - k\,a.c = 0$$

We know that $a.a = |a|^2$ $\quad c.c = |c|^2$

$$a.c = 2|a|^2\cos\theta$$
$$(1-k)\,2|a|^2\cos\theta - |a|^2 + k(1-k)|c|^2 - k[2|a|^2\cos\theta] = 0$$
$$(1-2k)\,2|a|^2\cos\theta - |a|^2 + k(1-k)\,4|a|^2 = 0$$
$$(1-2k)\,2|a|^2\cos\theta - |a|^2[1-4k+4k^2] = 0$$

$$(1-2k)\,2|a|^2\cos\theta - |a|^2(1-2k)^2 = 0$$

$$|a|^2(1-2k)\left[2\cos\theta - 1 + 2k\right] = 0$$

$$|a|^2(1-2k)(2\cos\theta - (1-2k)) = 0$$

(C) By result of Part b.

$|a| \neq 0 \qquad 1-2k = 0 \qquad k = \frac{1}{2}$ ✓

$$2\cos\theta - (1-2k) = 0$$

$$2\cos\theta - 1 + 2k = 0$$

$$2k = 1 - 2\cos\theta$$

$$k = \frac{1-2\cos\theta}{2}$$

$$k = \frac{1}{2} - \cos\theta$$

$$0 \leq \frac{1}{2} - \cos\theta \leq 1$$

$$-\frac{1}{2} \leq -\cos\theta \leq \frac{1}{2}$$

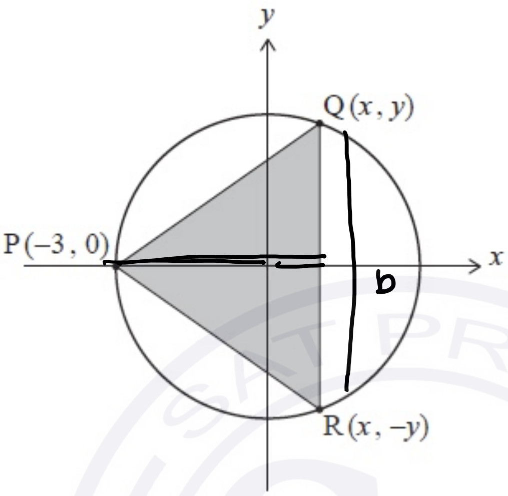
$$\frac{1}{2} \leq \cos\theta \leq -\frac{1}{2}$$

$$\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q10

A circle with equation $x^2+y^2=9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $P(-3, 0)$, $Q(x, y)$ and $R(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



$b = 2y$

$h = x+3$

(a) For point Q, show that $y=\sqrt{9-x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

(c) Show that $\dfrac{dA}{dx}=\dfrac{9-3x-2x^2}{\sqrt{9-x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

Sol (a) $x^2+y^2=9$

$y^2 = 9-x^2$

$y = \sqrt{9-x^2}$

(b) $A = \frac{1}{2} bh$

$= \frac{1}{2} 2y(x+3) = \frac{1}{\cancel{2}} \times \cancel{2} \times \sqrt{9-x^2} \times (x+3)$

$A = x\sqrt{9-x^2} + 3\sqrt{9-x^2}$

(c) $\dfrac{dA}{dx} = x \cdot \frac{1}{\cancel{2}}(9-x^2)^{-1/2}(-\cancel{2}x) + \sqrt{9-x^2} + 3\,\frac{1}{2}(9-x^2)^{-1/2} \times -2x$

$= \dfrac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} + \dfrac{-3x}{\sqrt{9-x^2}}$

$$= \frac{-x^2 + 9 - x^2 - 3x}{\sqrt{9-x^2}}$$

$$= \frac{9 - 3x - 2x^2}{\sqrt{9-x^2}}$$

(d) $\dfrac{dA}{dx} = 0$

$9 - 3x - 2x^2 = 0$

$9 - 6x + 3x - 2x^2 = 0$

$3(3-2x) + x(3-2x) = 0$

$(3-2x)(3+x) = 0$

$\underline{x = 3/2} \quad x = -3$

$y = -\sqrt{9-x^2} = -\sqrt{9 - \frac{9}{4}} = -\sqrt{\frac{27}{4}}$

$= -\dfrac{3\sqrt{3}}{2}$

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q11

Consider the complex number $u = -1 + \sqrt{3}\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2e^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

Sol (a) $u = -1 + \sqrt{3}i$

$|u| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\operatorname{Arg} u = \theta = \tan^{-1} -\sqrt{3}$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$u = 2e^{i\frac{2\pi}{3}}$

(b) (i) $u^n = 2^n e^{i\frac{2\pi}{3}n}$

$= 2^n\left[\cos\frac{2\pi}{3}n + i\sin\frac{2\pi}{3}n\right]$

$n = 3$

$\cos\frac{2\pi}{3} \times 3 = \cos 2\pi$ (Real)

$\sin\frac{2\pi}{3} \times 3 = \sin 2\pi$

$= 0$ (No imaginary part)

So when $n = 3$ the u^n would be real only.

b(ii) In part b(i) $n = 3$

So $$u^n = 2^n e^{i\frac{2\pi}{3}n}$$

$$u^3 = 2^3 e^{i\frac{2\pi}{3}n} \qquad \cos\frac{2\pi}{3}\times 3 = 1$$

$$= 8$$

c(i) $Z_1 = -1+\sqrt{3}i$

$Z_2 = -1-\sqrt{3}i$ (by. Conjugate root theorem)

let $z_3 = c$

Sum of root $= -5$

$$-1+\cancel{\sqrt{3}i}+c-1-\cancel{\sqrt{3}i} = -5$$

$$c-2 = -5$$

$$c = -5+2 = -3$$

hence roots would be

$-1+\sqrt{3}i$, $-1-\sqrt{3}i$ and -3

c(ii) Compare

$z^3+5z^2+10z+12 = 0$ and

$1+5w+10w^2+12w^3$

$z = \frac{1}{w} \quad \therefore \quad w = \frac{1}{z}$

hence

$$w = -\frac{1}{3}, \frac{1}{-1\pm\sqrt{3}i}$$

(d) $z^2 = 2z^*$

let $z = a + bi$

$$(a+bi)^2 = 2(a-bi)$$

$$\underline{a^2 - b^2} + 2abi = \underline{2a} - 2bi$$

✓ $a^2 - b^2 = 2a$ (Real part)

$2ab = -2b$ (Imaginary part)

$2ab + 2b = 0$

$2b(a+1) = 0$

$2b = 0$ ✓ $b = 0$

$a^2 = 2a$

$a^2 - 2a = 0$

$a(a-2) = 0$

$a = 0, 2$

$a = 2$ (Real)

$a + 1 = 0$

$a = -1$

$1 - b^2 = -2$

$b^2 = 3$

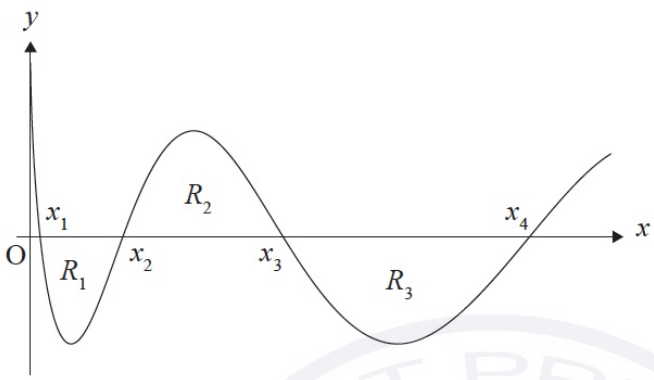
$b = \pm\sqrt{3}$

$b = \pm\sqrt{3}$ (complex root $-1 \pm \sqrt{3}i$)

Problem - M23/5/MATHX/HP1/ENG/TZ2/XX/Q12

(a) By using an appropriate substitution, show that $\int \cos\sqrt{x}\,\mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos\sqrt{x}$ for $x \ge 0$.



The curve intersects the x-axis at x_1, x_2, x_3, x_4, $\ldots$.

The nth x-intercept of the curve, x_n, is given by $x_n = \dfrac{(2n-1)^2\pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for x_{n+1}. [1]

The regions bounded by the curve and the x-axis are denoted by R_1, R_2, R_3, $\ldots$, as shown on the above diagram.

(c) Calculate the area of region R_n.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the x-axis, R_1, R_2, R_3, $\ldots$, form an arithmetic sequence. [3]

Sol (a) $y = \cos\sqrt{x}$

$\int \cos\sqrt{x}\,dx$ Let $t = \sqrt{x}$

$t^2 = x$

$2t\,dt = dx$

$\int 2t\cos t\,dt$

$2\int t\cos t\,dt$ $t = u$ $v' = \cos t$

$u' = 1$ $v = \sin t$

$\int uv' = uv - \int u'v$

$$2\left[t\sin t - \int 1 \sin t\, dt\right]$$

$$2\left[t\sin t - (-\cos t)\right] + c$$

$$2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$$

(b)

$$x_n = \frac{(2n-1)^2\pi^2}{4}$$

$$x_{n+1} = \frac{(2(n+1)-1)^2\pi^2}{4}$$

$$= \frac{(2n+1)^2\pi^2}{4}$$

(c)

$$A = \int_{x_n}^{x_{n+1}} \cos\sqrt{x}\, dx$$

$$= \int_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}} \cos\sqrt{x}\, dx$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}$$

$$= 2\left[\frac{(2n+1)\pi}{2}\sin\frac{(2n+1)\pi}{2} + \cos(2n+1)\frac{\pi}{2} - \frac{(2n-1)}{2}\sin\frac{(2n-1)\pi}{2} - \cos(2n-1)\frac{\pi}{2}\right]$$

$$= 2\left[(-1)^n\frac{(2n+1)\pi}{2} + 0 - \frac{(2n-1)}{2}(-1)^{n+1} + 0\right]$$

$$= 2\left[(-1)^n\frac{(2n+1)}{2}\pi + (-1)^n\frac{(2n-1)\pi}{2}\right]$$

$$= \cancel{x}\left[\frac{(-1)^n}{\cancel{x}}\left[2n\pi + \cancel{\pi} + 2n\pi - \cancel{\pi}\right]\right]$$

$$= |4n\pi\,(-1)^n|$$

$$= 4n\pi$$

(d) $A_n = 4n\pi$

$$A_{n+1} = 4(n+1)\pi$$

$$= 4n\pi + 4\pi$$

$$d = A_{n+1} - A_n$$

$$= \cancel{4n\pi} + 4\pi - \cancel{4n\pi}$$

$$= 4\pi$$

So as difference is constant hence area of regions R_1, R_2, R_3 - - - - are in arithmetic sequence