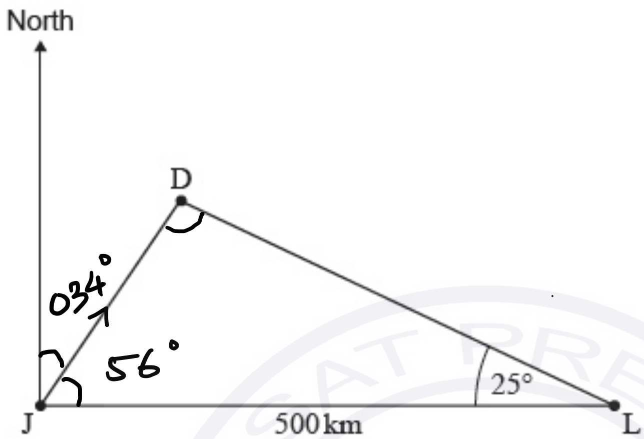
Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q1

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram.
Lucknow lies $500\,\mathrm{km}$ directly east of Jaipur, and $\mathrm{J\hat{L}D} = 25°$.

diagram not to scale



The bearing of D from J is 034°.

(a) Find $\mathrm{J\hat{D}L}$. [2]

(b) Find the distance between Lucknow and Delhi. [3]

Sol (a) $\angle DJL = 90 - 34° = 56°$

$\angle JDL = 180 - (56 + 25)$

$= 99°$

(b) By sine rule

$$\frac{DL}{\sin 56°} = \frac{500}{\sin 99°}$$

$$DL = \frac{500 \times \sin 56°}{\sin 99°}$$

$$= 419.686$$

$$\approx 420\text{ km}.$$

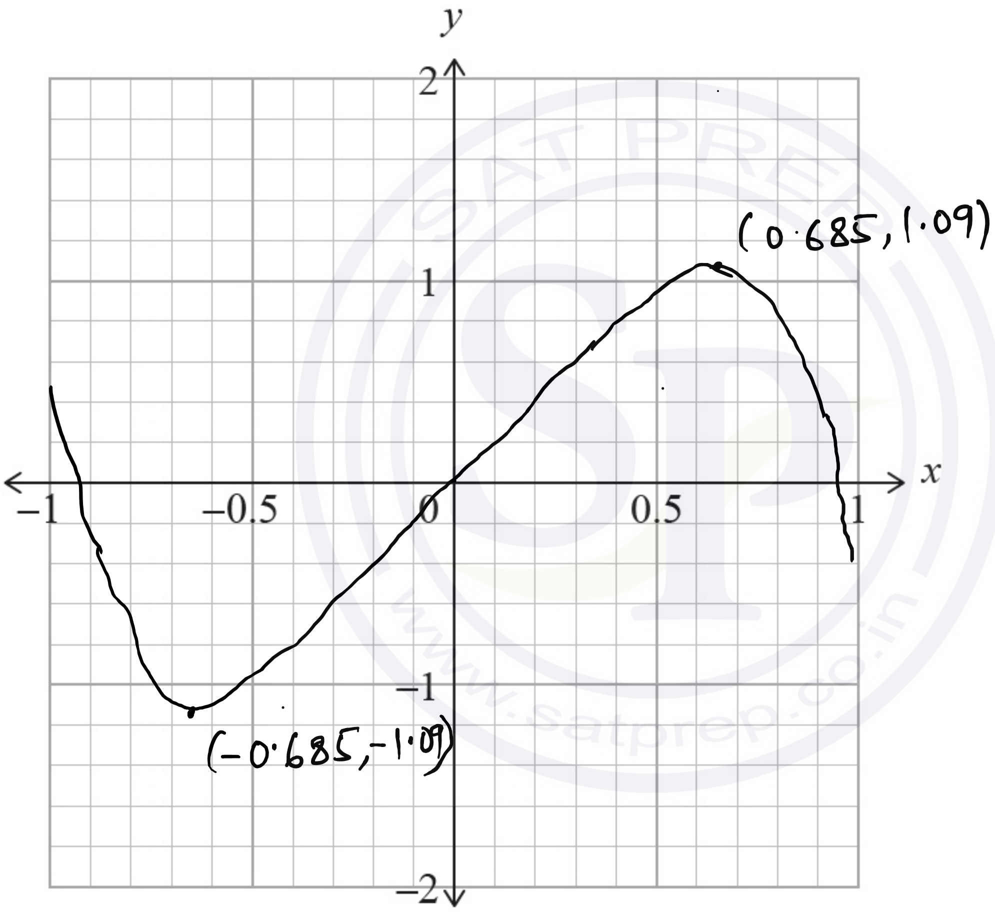
Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q2

[Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \le x \le 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



Sol (a) $(f \circ g)(x) = f(g(x))$

$= f(\tan x)$

$= 2\tan x - (\tan x)^3$

(b) on graph

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q3

[Maximum mark: 7]

The total number of children, y, visiting a park depends on the highest temperature, T, in degrees Celsius (°C). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \leq T \leq 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°\mathrm{C}$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, x. The following table shows the data collected on five different days.

Total number of children (y)	81	175	202	346	360
Ice creams sold (x)	15	27	23	35	46

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are y children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°\mathrm{C}$. [2]

Sol (a) $y = -0.6(25)^2 + 23(25) + 110$

$= 310$

(b) $x = 0.0935y + 7.43$

(c) $x = 0.0935(310) + 7.43$

$= 36.41$

≈ 36

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q4

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, D mm, are normally distributed with mean 32 and standard deviation σ. The interquartile range of the diameters is 0.28.

Find the value of σ.

Sol

$D \sim N(32, \sigma^2)$

$P(D < Q_3) = 0.75$

$$\frac{Q_3 - 32}{\sigma} = \text{InvN}(0.75)$$

$Q_3 = 32 + \text{InvN}(0.75)\sigma$

$Q_3 = 32 + 0.67449\sigma$

$P(D < Q_1) = 0.25$

$$\frac{Q_1 - 32}{\sigma} = \text{INVN}(0.25)$$

$Q_1 = 32 + \text{INVN}(0.25)\sigma$

$Q_1 = 32 - 0.67449\sigma$

$Q_3 - Q_1 = 0.28$

$2 \times 0.67449\sigma = 0.28$

$$\sigma = \frac{0.28}{2 \times 0.67449} = 0.2075$$

$= 0.208$

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q5

The coefficient of x^6 in the expansion of $(ax^3+b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3+b)^{10}$ is 2880.

Find the value of a and the value of b, where $a, b > 0$.

Sol

$${}^{8}C_r (ax^3)^{8-r} b^r \qquad x^6$$

$$x^{24-3r} \qquad x^6$$

$$24-3r=6$$

$$3r=18$$

$$r=6$$

$${}^{8}C_6 a^2 x^6 b^6$$

$$28a^2b^6 = 448 \quad \text{— (i)}$$

$${}^{10}C_r (ax^3)^{10-r} b^r \qquad x^6$$

$$30-3r=6$$

$$3r=24$$

$$r=8$$

$${}^{10}C_8 a^2 x^6 b^8$$

$$45a^2b^8 = 2880 \quad \text{— (ii)}$$

by solving eq(ii) & (i)

$$\frac{45a^2b^8}{28a^2b^6} = \frac{2880}{448}$$

$$b^2 = \frac{2880 \times 28}{45 \times 448} = 4$$

$$\boxed{b=2}$$

by eq(i)

$$28 \times a^2 \times 2^6 = 448$$

$$a^2 = \frac{448}{28 \times 64} = \frac{1}{4}$$

$$a = \frac{1}{2}$$

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q6

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of n that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to z on the Argand diagram that results in z^{10}. [3]

Sol (a)

$$z^n = \left(\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}\right)^n$$

$$= \cos\frac{11\pi}{18}n + i\sin\frac{11\pi}{18}n$$

$$= e^{\frac{11\pi}{18}ni}$$

$$\theta = \frac{3}{2}\pi \text{ or } \frac{7\pi}{2} \text{ or } \frac{11\pi}{2}$$

$$\frac{11\pi}{18}n = \frac{11\pi}{2}$$

$$n = 9$$

(b)

$$z^{10} = e^{\frac{11\pi}{18}\times 10}$$

$$= e^{\frac{110\pi}{18}} = e^{\frac{55\pi}{9}} = e^{\frac{\pi}{9}}$$

$$\operatorname{Arg}(z^{10}) - \operatorname{Arg}(z) = \frac{\pi}{9} - \frac{11\pi}{18} = -\frac{\pi}{2}$$

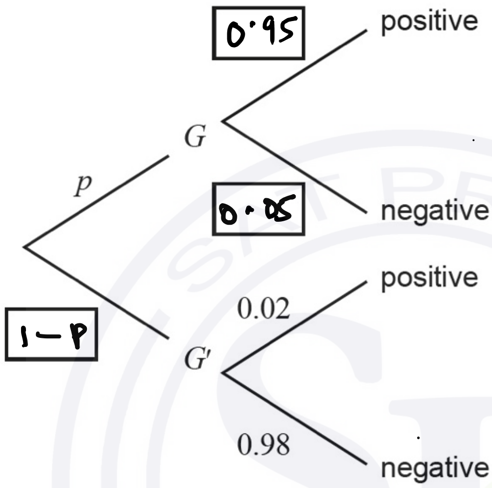
So this is rotation by $\frac{3}{2}\pi$ or $-\frac{\pi}{2}$ about origin.

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q7

A new test has been developed to identify whether a particular gene, G, is present in a population of parrots. The test returns a correct positive result 95% of the time for parrots with the gene, and a false positive result 2% of the time for parrots without the gene.

The proportion of the population with the gene is p.

(a) Complete the tree diagram below. [2]



(b) A random sample of the population was tested. It was found that 150 tests returned a positive result. Out of the 150 parrots with a positive test result, 18 did not actually have the gene. Find an estimate for p. [4]

Sol

$$P(G \mid \text{positive}) = \frac{P(G \cap \text{positive})}{P(\text{positive})}$$

$$1 - \frac{18}{150} = \frac{p \times 0.95}{p \times 0.95 + (1-p)\,0.02}$$

$$\frac{132}{150} = \frac{0.95p}{0.95p + 0.02 - 0.02p}$$

$$132(0.93p + 0.02) = 0.95p \times 150$$

$$122.76p + 2.64 = 142.5p$$

$$2.64 = 19.74p$$

$$p = 0.1337$$

$$\approx 0.134$$

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q8

The angle between a line and a plane is α, where $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

$\frac{z-5}{-1}$

Find the value of α.

Sol Normal vector of plane is $\begin{pmatrix} 4 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$

Direction vector of line is $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

Angle between line and plane

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \frac{12 + 2\cos\alpha - \sin\alpha}{\sqrt{4^2 + \cos^2\alpha + \sin^2\alpha}\sqrt{3^2 + 2^2 + (-1)^2}}$$

$$\sin\alpha = \frac{12 + 2\cos\alpha - \sin\alpha}{\sqrt{17}\sqrt{14}}$$

$$\sqrt{17}\sqrt{14}\sin\alpha = 12 + 2\cos\alpha - \sin\alpha$$

$$\alpha = 0.932$$

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q9

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

Sol Assume

$$p^2 - 8q - 11 = 0$$

$$p^2 = 8q + 11$$

We conclude that

p^2 is odd

$\therefore$ p is odd

$$p = 2k + 1$$

$$(2k+1)^2 = 8q + 11$$

$$4k^2 + 4k + 1 = 8q + 11$$

$$4k^2 + 4k = 8q + 10$$

$$2k^2 + 2k = 4q + 5$$

So we can say
a contradiction as LHS even and RHS odd
therefor if $p, q \in \mathbb{Z}$ then

$$p^2 - 8q - 11 \neq 0$$

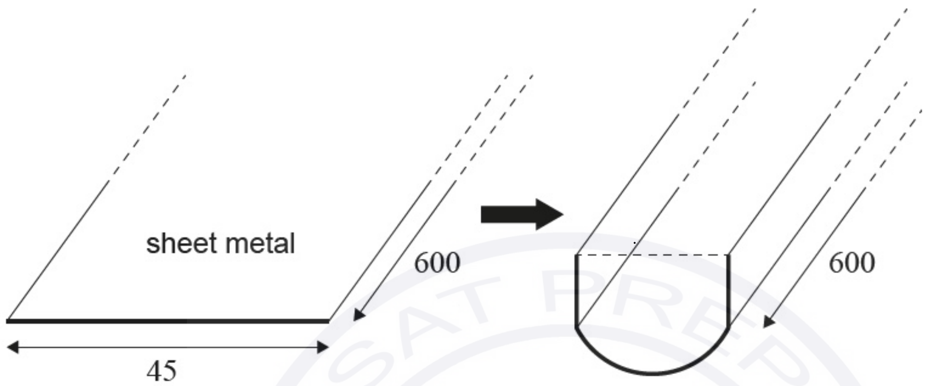
hence proved.

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q10

An engineer is designing a gutter to catch rainwater from the roof of a house.

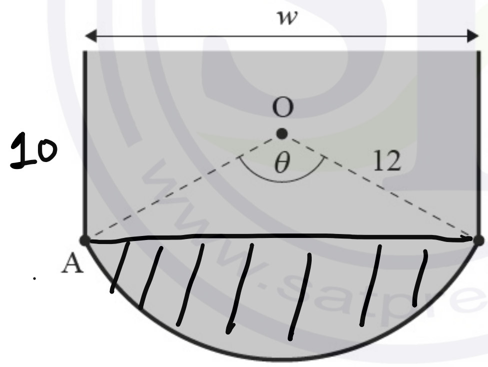
The gutter will be open at the top and is made by folding a piece of sheet metal $45\,\text{cm}$ wide and $600\,\text{cm}$ long.

diagram not to scale



The cross-section of the gutter is shaded in the following diagram.

diagram not to scale



Arc AB $= 45-20$
$= 25$

The height of both vertical sides is $10\,\text{cm}$. The width of the gutter is $w\,\text{cm}$.

Arc AB lies on the circumference of a circle with centre O and radius $12\,\text{cm}$.

Let $A\hat{O}B = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3, of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000,\ t \geq 0.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Sol (a) Arc length $= r\theta$

$25 = 12\theta$

$\theta = \frac{25}{12} = 2.08$

(b) Area of cross section

$AB = \sqrt{12^2 + 12^2 - 2 \times 12^2 \cos 2.08}$

$= 20.7$ cm

Shaded part

$= \frac{1}{2} \times 12^2 \times 2.08 - \frac{1}{2} \times 12^2 \times \sin 2.08$

$= 86.9\ cm^2$

Area of cross section

$= 10 \times 20.7 + 86.9$

$= 293.87 \approx 294\ cm^2$

(c) Volume of gutter $= 294 \times 600$

$= 176400 \text{ cm}^3$

Volume of water in 60 sec period

$$= \int_0^{60} R'(t)\,dt$$

$$= \int_0^{60} 50\cos\left(\frac{2\pi t}{5}\right) + 3000\,dt$$

$$= 180000 \text{ cm}^3$$

Volume of water > Volume of gutter hence gutter will overflow in this storm.

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q11

A continuous random variable, X, has a probability density function defined by

$$f(x)=\begin{cases}\dfrac{6}{\pi\sqrt{16-x^2}}, & 0\le x\le 2\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X<0.5)$. [2]

A laboratory trial may require up to 2 millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where X is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X<0.5$.

(c) Determine the least number of trials required to be 99% sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider n trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of n such that the probability of three consecutive successful trials is more than 0.05. [6]

Sol (a)

$$E(X)=\int x\,f(x)\,dx$$

$$=\int_0^2 \frac{x\cdot 6}{\pi\sqrt{16-x^2}}\,dx$$

$$=\frac{6}{\pi}\int_0^2\frac{x}{\sqrt{16-x^2}}\,dx$$

Let $16-x^2=t$ $\qquad x=0 \quad t=16$

$-2x\,dx=dt \qquad x=2 \quad t=12$

$x\,dx=-\frac{1}{2}dt$

$$\frac{6}{\pi}\times\frac{-1}{2}\int_{16}^{12}\frac{1}{\sqrt{t}}\,dt \qquad \frac{t^{-1/2+1}}{-1/2+1}=\frac{t^{1/2}}{1/2}$$

$$-\frac{3}{\pi}\left[\frac{\sqrt{t}}{1/2}\right]_{16}^{12}$$

$$-\frac{6}{\pi}\left[\sqrt{12}-\sqrt{16}\right]$$

$$-\frac{6}{\pi}\left[2\sqrt{3}-4\right]$$

$$\frac{12}{\pi}\left[2-\sqrt{3}\right]$$

(b) $P(X<0.5)=\int_{0}^{0.5} f(x)\,dx$

$$=\int_{0}^{0.5}\frac{6}{\pi\sqrt{16-x^2}}\,dx$$

$$=0.239$$

(c) $X \sim B(n, 0.239)$

$P(X\geq 1) = 1 - P(X=0) \geq .99$

$= 1 - {}^{n}C_{0}(0.239)^{0}(1-0.239)^{n})$

$1-(1-0.239)^{n}) \geq .99$

$n \geq 16.8612$

$n \approx 17$

(d) $X \sim B(10, 0.239)$

$P(X=3) = 0.242$

(e) $10 - 2 = 8$

(f) (i) $n - 2$

(ii) ${}^{n}C_3$ (3 way of success in n trials)

$$\frac{n-2}{{}^{n}C_3} > 0.05$$

$$\frac{n-2}{\frac{n!}{3!\,(n-3)!}} > 0.05$$

$$\frac{\cancel{(n-2)} \times 6 \times \cancel{(n-3)!}}{n \times (n-1) \times \cancel{(n-2)} \times \cancel{(n-3)!}} > 0.05$$

$$\frac{6}{n \times (n-1)} > 0.05$$

$n = 11$

Problem - M23/5/MATHX/HP2/ENG/TZ1/XX/Q12

Consider the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2+1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

(ii) Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{d^3y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y.

(ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2-1}{x^2+1} \equiv 1 - \frac{2}{x^2+1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2+1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \le x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y. [2]

Sol (a) $\frac{dy}{dx} = \frac{y(x^2-1)}{x^2+1}$

$x_0 = 0 \quad y_0 = 3$

step length $= 0.03$

$x_{n+1} = x_n + \text{step length}$

$y_{n+1} = y_n + \text{step length} \times f(x_n, y_n)$

$x_1 = x_0 + 0.03$

$x_1 = 0 + 0.03 = 0.03$

$x_2 = 0.03 + 0.03 = 0.06$

$x_3 = 0.06 + 0.03 = 0.09$

$x_4 = 0.09 + 0.03 = 0.12$

$y_1 = y_0 + 0.03 \times [f(x_0\, y_0)]$

$y_1 = 3 + 0.03 \times 3\frac{(0^2-1)}{0^2+1} = 2.91$

$y_2 = y_1 + 0.03 \times f(x_1, y_1)$

$= 2.91 + 0.03 \times 2.91\left[\frac{0.03^2-1}{0.03^2+1}\right]$

$= 2.822857 = 2.82285$

$y_3 = 2.82285 + 0.03 \times 2.82285\left[\frac{0.06^2-1}{0.06^2+1}\right]$

$= 2.73877$

$y_4 = 2.73877 + 0.03 \times 2.73877\left[\frac{0.09^2-1}{0.09^2+1}\right]$

$= 2.65792$

$y_5 = 2.65792 + 0.03 \times 2.65792\left[\frac{0.12^2-1}{0.12^2+1}\right]$

$= 2.58044$

$\therefore y(0.15) = 2.58044$

(b) (i) $\dfrac{dy}{dx} = \dfrac{yx^2 - y}{x^2+1}$

$x=0 \quad \dfrac{dy}{dx} = \dfrac{0-3}{0-1} = -3$

(ii) $\dfrac{dy}{dx} = \dfrac{yx^2-y}{x^2+1}$

differentiate using quotient Rule

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)\left(y.2x + x^2\frac{dy}{dx} - \frac{dy}{dx}\right) - y(x^2-y)(2x)}{(x^2+1)^2}$$

$x=0 \quad y=3$

$$= \frac{(0^2+1)(3.2(0) + 0^2(-3) - \underline{(-3)} - (-3)(0^2-(-3))(2\times 0)}{(0^2+1)^2}$$

$\dfrac{d^2y}{dx^2} = 3$

c (i) $y = y(0) + x\,y'(0) + \dfrac{x^2}{2!}y''(0) + \dfrac{x^3}{3!}y'''(0)$

$= 3 + x(-3) + \dfrac{x^2}{2!}3 + \dfrac{x^3}{3!}9$

$= 3 - 3x + \dfrac{3}{2!}x^2 + \dfrac{9}{3!}(x)^3$

(ii) $y(0{\cdot}15) = 3 - 3(0{\cdot}15) + \dfrac{3}{2}(0{\cdot}15)^2 + \dfrac{3}{2}(0{\cdot}15)^3$

$= 2{\cdot}58881$

d (i) $\dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

$\dfrac{dy}{dx} = \dfrac{x^2y-y}{x^2+1} = \dfrac{y(x^2-1)}{x^2+1}$

$\displaystyle\int \frac{dy}{y} = \int \frac{x^2-1}{x^2+1}dx = \int 1 - \frac{2}{x^2+1}\,dx$

$\ln y = x - 2\tan^{-1}x + c$

$x=0 \quad y=3$

$\ln 3 = 0 - 2\tan^{-1}0 + c$

$c = \ln 3$

$\ln y = x - 2\tan^{-1}x + \ln 3$

$$y = e^{[x - 2\tan^{-1} x + \ln 3]}$$

d(ii) $y(0.15) = e^{[0.15 - 2\tan^{-1}(0.15) + \ln 3]}$

$= 2.58786$

(e) the graph $y = f(x)$ is concave $\frac{d^2y}{dx^2} > 0$ hence the tangents used in euler' method gives underestimate so approximate value of y when $x = 0.15$ is less than the actual value of y.