Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q1

A botanist is conducting an experiment which studies the growth of plants.

The heights of the plants are measured on seven different days.

The following table shows the number of days, d, that the experiment has been running and the average height, h cm, of the plants on each of those days.

Number of days (d)	2	5	13	24	33	37	42
Average height (h)	10	16	30	59	76	79	82

The value of Pearson's product-moment correlation coefficient, r, for this data is 0.991, correct to three significant figures.

(a) The regression line of h on d for this data can be written in the form $h = ad + b$.

Find the value of a and the value of b. [2]

(b) Use your regression line to estimate the average height of the plants when the experiment has been running for 20 days. [2]

Sol (a) $a = 1.93$

$b = 7.22$

(b) $h = 1.93d + 7.22$

$d = 20$

$h = 45.868$

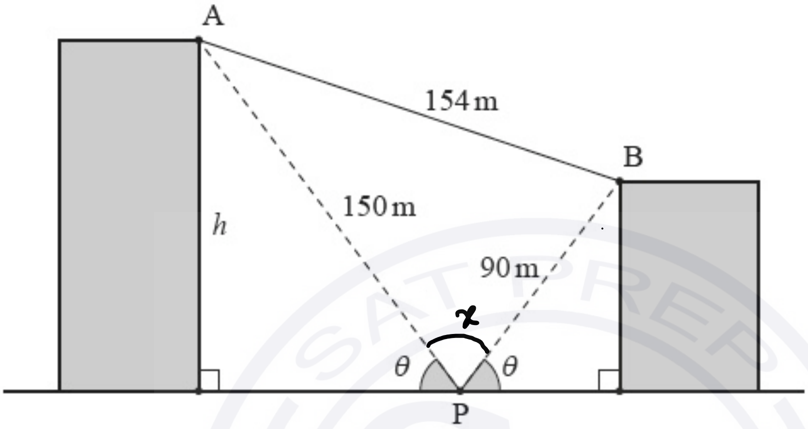
$= 45.9$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q2

The following diagram shows two buildings situated on level ground.

From point P on the ground directly between the two buildings, the angle of elevation to the top of each building is θ.

diagram not to scale



The distance from point P to point A at the top of the taller building is 150 metres.

The distance from point P to point B at the top of the shorter building is 90 metres.

The distance between A and B is 154 metres.

Find the height, h, of the taller building.

<u>Sol</u> $\triangle APB$

$$AB^2 = AP^2 + BP^2 - 2\,AP \times BP \times \cos x$$

$$\cos x = \frac{AP^2 + BP^2 - AB^2}{2\,AP \times BP}$$

$$\cos x = \frac{150^2 + 90^2 - 154^2}{2 \times 150 \times 90}$$

$$x = \cos^{-1}\left[\frac{150^2 + 90^2 - 154^2}{2 \times 150 \times 90}\right]$$

$$= 75.2^\circ$$

$$\theta = \frac{180 - 75.2}{2} = 52.38$$

$$= 52.4$$

$$h = 150 \sin 52.4$$

$$h = 118.811$$

$$\approx 119 \text{ m}$$

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Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q3

The weights, W grams, of bags of rice packaged in a factory can be modelled by a normal distribution with mean 204 grams and standard deviation 5 grams.

(a) A bag of rice is selected at random.

Find the probability that it weighs more than 210 grams. [2]

According to this model, 80% of the bags of rice weigh between w grams and 210 grams.

(b) Find the probability that a randomly selected bag of rice weighs less than w grams. [2]

(c) Find the value of w. [2]

(d) Ten bags of rice are selected at random.

Find the probability that exactly one of the bags weighs less than w grams. [2]

Sol

$$W \sim N(204, 5^2)$$

(a) $P(W > 210) = 0.1150$

$$P(w < W < 210) = 0.8$$

$$P(W < 210) - P(W < w) = 0.8$$

$$1 - P(W > 210) - P(W < w) = 0.8$$

$$1 - 0.1150 - P(W < w) = 0.8$$

(b) $P(W < w) = 1 - 0.1150697 - 0.8$

$$P(W < w) = 0.08493$$

(c) $$\frac{w - 204}{5} = \text{INVN}(0.08493)$$

$$w = 204 + 5 \times \text{InvN}(0.08493)$$

$$= 204 + 5 \times -1.37265$$

$$= 197.137$$

$$= 197$$

(d) $X \sim B(10, 0.08493)$

$P(X=1) = 0.382$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q4

The expansion of $(x+h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + \dots + h^8$, where $a, b, c, d, \dots \in \mathbb{R}$.

Given that the coefficients, a, b and d, are the first three terms of a geometric sequence, find the value of h.

Sol $(x+h)^8 = {}^8C_0 x^8 h^0 + {}^8C_1 x^7 h + {}^8C_2 x^6 h^2 + {}^8C_3 x^5 h^3 + {}^8C_4 x^4 h^4 + \dots h^8$

Compare with $x^8 + ax^7 + bx^6 + cx^5 + dx^4$

$x^7 \quad a = {}^8C_1 h \quad$ 1st term

$x^6 \quad b = {}^8C_2 h^2 \quad$ 2nd term

$x^5 \quad c = {}^8C_3 h^3$

$x^4 \quad d = {}^8C_4 h^4 \quad$ 3rd term

$$b^2 = ad$$

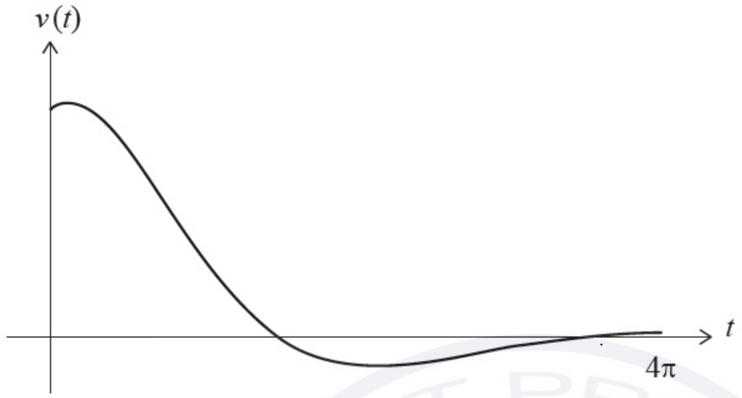
$$({}^8C_2 h^2)^2 = {}^8C_1 h \times {}^8C_4 h^4$$

$$28^2 \cancel{h^4} = 8h \times 70 \cancel{h^4}$$

$$h = \frac{28 \times 28}{8 \times 70} = 1.4$$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q5

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of v is shown in the following diagram.



Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1. [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2. [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

Sol (a) $t_1 = 0.395$

(b) $t_2 = 10.9956$

$= 11$

(c) distance $= \displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$

$$= \int_{0.395}^{10.9956} \left|4e^{-\frac{x}{3}}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right| dx$$

$= 7.83$ m

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q6

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let L be the line of intersection of Π_1 and Π_2.

(a) Verify that a vector equation of L is $r = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]

(b) Find the coordinates of the point P on L that is nearest to the origin. [5]

Sol (a) let us consider a point in terms of λ on line L

$$(\lambda, 2-2\lambda, 4-2\lambda)$$

Π_1 $\quad 2(\lambda) - (2-2\lambda) + 2(4-2\lambda)$

$$2\lambda - 2 + 2\lambda + 8 - 4\lambda$$

$$= 6$$

Π_2 $\quad 4(\lambda) + 3(2-2\lambda) - (4-2\lambda)$

$$4\lambda + 6 - 6\lambda - 4 + 2\lambda$$

$$= 2$$

(b) the position vector for point P nearest to origin is perpendicular to direction vector of line L.

$$\begin{pmatrix} \lambda \\ 2-2\lambda \\ 4-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 0$$

$$\lambda + (-2)(2-2\lambda) + (-2)(4-2\lambda) = 0$$

$$\lambda - 4 + 4\lambda - 8 + 4\lambda = 0$$

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$9\lambda = 12$

$\lambda = \frac{4}{3}$

$P\left(\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$

$2 - \frac{8}{3}$

$4 - 2 \times \frac{4}{3}$

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Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q7

A function f is defined as $f(x) = \arctan(x-2)$, where $2 \le x \le 2+\sqrt{3}$.

The region bounded by the curve, the y-axis, the x-axis and the line $y = \frac{\pi}{3}$ is rotated $360°$ about the y-axis to form a solid of revolution.

Find the volume of the solid.

Sol

$$f(x) = \text{Arc}\tan(x-2)$$

$$V = \pi \int_{x=a}^{b} [f^{-1}(x)]^2 \, dx$$

let $y = \text{Arc}\tan(x-2)$

$$\tan y = x - 2$$

$$x = \tan y + 2$$

$$f^{-1}(x) = \tan x + 2$$

$$2 = \tan x + 2$$

$$\tan x = 0$$

$$x = 0$$

$$2+\sqrt{3} = \tan x + 2$$

$$x = \tan^{-1}\sqrt{3}$$

$$= \pi/3$$

$$V = \pi \int_{0}^{\pi/3} (\tan x + 2)^2 \, dx$$

$$= 24.02$$

$$= 24.0$$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q8

A function g is defined by $g(x)=\frac{2x-5}{x^2-3}$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$.

(a) Determine the range of g. [4]

A function h is defined by $h(x)=g(|x|)\cos t$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$ and t is a constant where $\frac{\pi}{2} < t \leq \pi$.

(b) Find the set of values of x such that $h(x) \leq 0$. [3]

Sol (a) $g(x)=\frac{2x-5}{x^2-3}$

local Minima $y = 1.43$

local Maxima $y = 0.232$

$g(x) \leq 0.232$ or $g(x) \geq 1.43$

(b) $\frac{2|x|-5}{x^2-3} \geq 0$ since $\cos t < 0$ for $\frac{\pi}{2} < t \leq \pi$

by graph we can say that set of values of x would be

$x \leq -\frac{5}{2}$ or $-\sqrt{3} < x < \sqrt{3}$ or $x \geq \frac{5}{2}$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q9

Let S be the set of 30 positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from S without replacement. He then adds them together and determines whether the sum is divisible by 3.

Determine the total number of selections Raghu can make to obtain a sum that is divisible by 3.

You may assume that order is not important, for example, $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{2, 1, 3\}, \{3, 1, 2\}, \{3, 2, 1\}$ are all considered to be the same selection.

Sol Now select 3 number from 30 without any restriction

$= {}^{30}C_3 = 4060$

Now we choose 2 numbers from a group of 3n

$= {}^{10}C_2 = 45$

Also we choose 1 number from a group of 3n-1

$= {}^{10}C_1 = 10$

So total No of way 3 number are choosen

$= 3! \times {}^{10}C_2 \times {}^{10}C_1$

$= 6 \times 45 \times 10 = 2700$

So total No of ways 3 numbers are choosen and their sum is divisible by 3

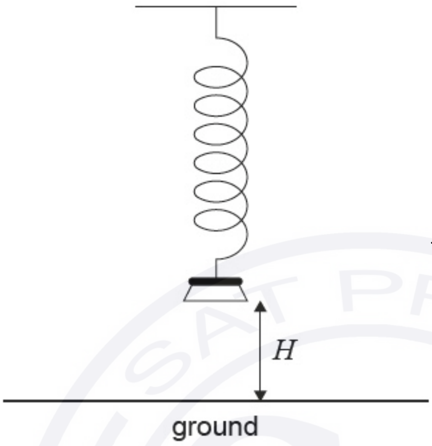
$= 4060 - 2700$

$= 1360$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q10

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

The height, H metres, of the base of the weight above the ground can be modelled by the function $H(t) = a\cos(7.8t) + b$, for a, $b \in \mathbb{R}$ and $0 \le t \le 10$, where t is the time in seconds after the weight is released.



(a) Find the period of the function. [2]

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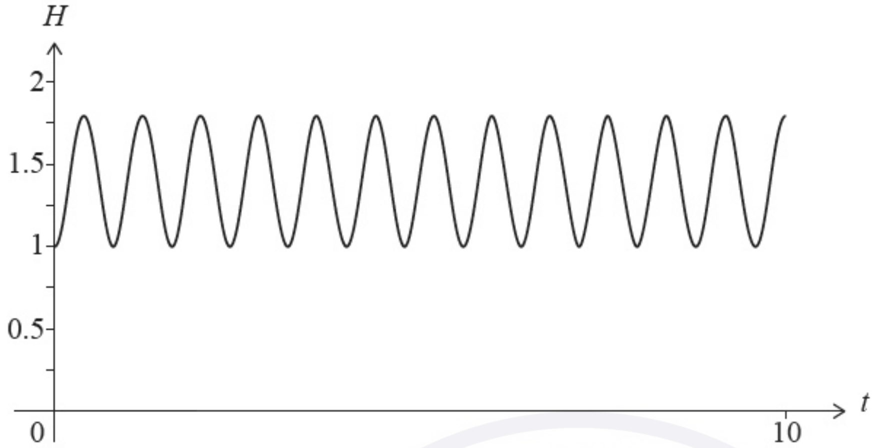
The weight is released when its base is at a minimum height of 1 metre above the ground, and it reaches a maximum height of 1.8 metres above the ground. The graph of H is shown in the following diagram.



(b) Find the value of

(i) a;

(ii) b. [3]

(c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]

(d) Find the first time that the base of the weight reaches a height of 1.5 metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

(e) Find the probability that the height of the base of the weight is greater than 1.5 metres at the time the picture is taken. [4]

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Sol (a) $H(t) = a\cos(7.8t) + b$

period $= \dfrac{2\pi}{b}$

$= \dfrac{2\pi}{7.8} = 0.806$

(b) (i) $\dfrac{\text{Max} - \text{Min}}{2} = \dfrac{1.8 - 1}{2} = -0.4 = a$

(ii) $\dfrac{\text{Max} + \text{Min}}{2} = \dfrac{1.8 + 1}{2} = 1.4 = b$

(c) So number of max height in five second $= 6$

(d) 0·234 sec

(e) Second line cross height 1·5 m.

$= 0·572$

So

P(height greater than 1·5)

$$= \frac{(0·572 - 0·234)\,6}{5}$$

$= 0·4056$

$= 0·406$

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Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q11

A game of chance involves drawing **two** balls at random out of a box without replacement.
The box initially contains r red balls and y yellow balls.

Let $\mathrm{P}(YY)$ represent the probability of drawing two yellow balls from the box without replacement.

Consider a version of this game where it is known that $\mathrm{P}(YY)=\frac{1}{3}$.

(a) Show that $2y^2-2(r+1)y+r-r^2=0$. [4]

(b) By solving the equation in part (a), show that $y=\frac{(r+1)+\sqrt{3r^2+1}}{2}$. [4]

(c) Find two pairs of values for r and y that satisfy the condition $\mathrm{P}(YY)=\frac{1}{3}$. [4]

Now consider a similar game of chance that involves drawing **three** balls out of a box without replacement. The box initially contains 10 red balls and y yellow balls.

Let $\mathrm{P}(YYY)$ represent the probability of drawing three yellow balls from the box without replacement.

(d) Find an expression for $\mathrm{P}(YYY)$ in terms of y. [3]

A yellow ball is added so that the box now contains 10 red balls and $(y+1)$ yellow balls.
The probability of drawing three yellow balls from the box without replacement is now twice the probability expressed in part (d).

(e) Find the initial number of yellow balls in the box. [5]

Sol (a) Box Contains total $= r+y$

Now drawing 1st yellow ball

$$P(Y)=\frac{y}{y+r}$$

Now drawing 2nd yellow ball

$$P(YY)=\frac{(y-1)\;y}{(y+r-1)(y+r)}$$

$$\frac{1}{3}=\frac{y^2-y}{(y+r-1)(y+r)}$$

$$(y+r-1)(y+r)=3y^2-3y$$

$$y^2+yr-y+yr+r^2-r=3y^2-3y$$

$$2y^2-2yr-2y+r-r^2=0$$

$$2y^2-2(r+1)y+r-r^2=0$$

(b) by using result of part (a) and quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{2(r+1) \pm \sqrt{4(r+1)^2 - 4 \times 2 \times (r - r^2)}}{4}$$

$$= \frac{2(r+1) \pm \sqrt{4r^2 + \cancel{8r} + 4 - \cancel{8r} + 8r^2}}{4}$$

$$= \frac{2(r+1) \pm 2\sqrt{3r^2+1}}{4}$$

$$= \frac{(r+1) \pm \sqrt{3r^2+1}}{2}$$

As $y, r \in Z^+$

hence

$$y = \frac{(r+1) + \sqrt{3r^2+1}}{2}$$

(c) As we know $r, y \in Z^+$

$\underline{r=1}$ $\quad y = \frac{(1+1) + \sqrt{3(1)^2+1}}{2} = \underline{2}$

$r = 2 \quad \times \quad y \notin Z^+$

$r = 3 \quad y = \frac{(2+1) + \sqrt{3(2)^2+1}}{2} =$

$r = 4 \quad y = \frac{(4+1) + \sqrt{3(4)^2+1}}{2}$

$= 6$

$r = 4 \quad y = 6$

(d) No. of Red balls $= 10$

No of yellow balls $= y$

Total No of balls $= 10 + y$

Now drawing 1^{st} ball

$$P(Y) = \frac{y}{10+y}$$

Now drawing 2^{nd} ball

$$P(YY) = \frac{(y-1)}{(10+y-1)} \frac{y}{(10+y)}$$

Now drawing 3^{r} ball

$$P(YYY) = \frac{(y-2)}{(10+y-2)} \frac{(y-1)}{(10+y-1)} \frac{y}{10+y}$$

$$= \frac{y(y-1)(y-2)}{(y+8)(y+9)(y+10)}$$

(e) Now one yellow ball added in box

So total NO of balls $= 10+y+1$

$$= 11+y$$

$$P(YYY) = \frac{(y+1)}{(y+11)} \frac{(y)}{(10+y)} \frac{(y-1)}{(9+y)}$$

$$\frac{2\cancel{y}\cancel{(y-1)}(y-2)}{(y+8)\cancel{(y+9)}\cancel{(y+10)}} = \frac{(y+1)\cancel{y}\cancel{(y-1)}}{(y+11)\cancel{(y+10)}\cancel{(y+9)}}$$

$$\frac{2(y-2)}{(y+8)} = \frac{(y+1)}{(y+11)}$$

$$2(y^2+11y-2y-22) = y^2+y+8y+8$$

$$2y^2+18y-44 = y^2+9y+8$$

$$y^2+9y-52=0$$

$$y=4$$

Problem - M23/5/MATHX/HP2/ENG/TZ2/XX/Q12

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0$, $y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of y when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \le x \le 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y, determine the x-coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0$, $y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y = mx$ where the exact value of m is to be determined. [6]

<u>Sol</u> (a) $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$ $\quad y_0 = 2 \quad x_0 = 1$

y when $x = 1.1$

step length $= 0.1$

$$y_1 = y_0 + \text{Step length} \times f(x_0, y_0)$$
$$= 2 + 0.1 \times \left[\frac{1^2 + 3(2)^2}{1 \times 2}\right]$$

$$y(1.1) = 2.65$$

(b) $$\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$$

So given eq is homogeneous

$$y = vx$$

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{x.vx}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v^2)}{\cancel{x^2}v} = \frac{1+3v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{v} - v = \frac{1+2v^2}{v}$$

by variable seperable method

$$\int \frac{v}{1+2v^2}\,dv = \int \frac{dx}{x}$$

let $1+2v^2 = t$

$$4v\,dv = dt$$

$$v\,dv = \frac{dt}{4}$$

$$\frac{1}{4}\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$y = vx$$

$$v = \frac{y}{x}$$

$$\frac{1}{4}\ln(1+2v^2) = \ln x + c$$

$$\frac{1}{4}\ln\left(1+2\frac{y^2}{x^2}\right) = \ln x + c$$

$x = 1$ $y = 2$

$$\frac{1}{4}\ln(1+8) = \ln 1 + c$$

$$\frac{1}{4}\ln(9) = c$$

$$\frac{1}{4}\ln\left(1+\frac{2y^2}{x^2}\right) = \ln x + \frac{1}{4}\ln 9$$

$$\ln\left(1+2\left(\frac{y}{x}\right)^2\right) = 4\ln x + \ln 9$$

$$\cancel{\ln}\left(1+2\left(\frac{y}{x}\right)^2\right) = \cancel{\ln}\, 9x^4$$

$$2\left(\frac{y}{x}\right)^2 = 9x^4 - 1$$

$$\left(\frac{y}{x}\right)^2 = \frac{9x^4-1}{2}$$

$$y = \pm x\sqrt{\frac{9x^4-1}{2}}$$

$y > 0 \quad y = x\sqrt{\dfrac{9x^4 - 1}{2}}$

(c) Substitute $x = 1.1$

$$y = 1.1\sqrt{\frac{9(1.1)^4 - 1}{2}}$$

$$= 2.714$$

(d) $\dfrac{d^2y}{dx^2} > 0$ for $1 \leq x \leq 1.1$

hence tangent drawn using euler's method gives an underestimate of the true value so the value of y when $x = 1.1$ is greater than the approximate value found in part (a).

(e) $x = 0.656$

$x = 0.656$

(f) $$\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$$

$\dfrac{d^2y}{dx^2} = 0 \quad \therefore \quad -x^4 + x^2y^2 + 6y^4 = 0$

$$-\left(\frac{x}{y}\right)^4 + \left(\frac{x}{y}\right)^2 + 6 = 0$$

let $\dfrac{y}{x} = m \qquad -\dfrac{1}{m^4} + \dfrac{1}{m^2} + 6 = 0$

$$6m^4 + m^2 - 1 = 0$$

let $m^2 = t \qquad 6t^2 + t - 1 = 0$

$$6t^2 + 3t - 2t - 1 = 0$$

$$3t(2t+1) - 1(2t+1) = 0$$

$$(3t - 1)(2t + 1) = 0$$

$m^2 = \frac{1}{3}$ $m^2 = -\frac{1}{2}$ ×

$m = \pm \frac{1}{\sqrt{3}}$

$y = \frac{1}{\sqrt{3}}x$ as $x > 0$ $y > 0$

so $m > 0$

$m = \frac{1}{\sqrt{3}}$

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