Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q1

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

Sol (a) $(g \circ f)(x) = g(f(x))$

$= g(x-3)$

$= (x-3)^2 + k^2$ ✓

(b) $(g \circ f)(2) = 10$

$(2-3)^2 + k^2 = 10$

$1 + k^2 = 10$

$k^2 = 9$

$k = 3, k = -3$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q2

Events A and B are such that $P(A) = 0.65$, $P(B) = 0.75$ and $P(A \cap B) = 0.6$.

(a) Find $P(A \cup B)$. [2]

(b) Hence, or otherwise, find $P(A' \cap B')$. [2]

Sol (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.65 + 0.75 - 0.6$

$= 0.8$

(b) $P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - 0.8$

$= 0.2$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q3

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

Sol (a) $S_4 = 40$

$40 = 16p - 4q$

$10 = 4p - q \quad \text{—(i)}$ ✓

$S_5 = 65$

$65 = 25p - 5q$

$13 = 5p - q \quad \text{—(ii)}$

by solving eq (i) & (ii)

$10 = 4p + 13 - 5p$

$-3 = -p$

$\underline{p = 3}$

$10 = 4 \times 3 - q$

$q = 12 - 10 = \underline{2}$

(b) $U_5 = S_5 - S_4$

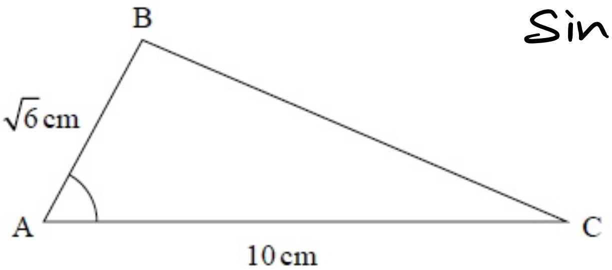
$= 65 - 40$

$= 25$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q4

In the following triangle ABC, $AB = \sqrt{6}$ cm, $AC = 10$ cm and $\cos B\hat{A}C = \frac{1}{5}$.

diagram not to scale



$\sin BAC = \sqrt{1-\left(\frac{1}{5}\right)^2}$

$= \frac{\sqrt{24}}{5}$

Find the area of triangle ABC.

Sol

$\text{Area} = \frac{1}{2} \times AB \times AC \times \sin BAC$

$= \frac{1}{\cancel{2}} \times \sqrt{6} \times \cancel{10} \times \frac{\sqrt{24}}{\cancel{5}}$

$= \sqrt{6 \times 24} = \sqrt{144}$

$= 12 \text{ cm}^2$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q5

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + \ldots + k^n x^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

Sol

$$(1+kx)^n = 1 + nkx + \frac{n(n-1)}{2!}k^2x^2$$

$$nk = 12$$

$$\frac{n(n-1)}{2}\cancel{k^2} = 28\cancel{k^2}$$

$$n^2 - n - 56 = 0$$

$$n^2 - 8n + 7n - 56 = 0$$

$$n(n-8) + 7(n-8) = 0$$

$$(n-8)(n+7) = 0$$

$$n = 8 \qquad n = -7$$

$$k = \frac{12}{8} = \frac{3}{2}$$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q6

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^+$.

Sol Base Case $n = 1$

$$5^2 - 2^3 = 25 - 8 = 17$$

So true for $n = 1$

Assume true for $n = k$ i.e.

$$5^{2k} - 2^{3k} = 17s \quad \text{where } s \in \mathbb{Z}$$

Consider $n = k+1$

$$5^{2(k+1)} - 2^{3(k+1)}$$

$$5^{2k} \cdot 25 - 2^{3k} \cdot 8$$

$$25(17s + 2^{3k}) - 8 \cdot 2^{3k}$$

$$25 \times 17s + 25 \cdot 2^{3k} - 8 \cdot 2^{3k}$$

$$17 \cdot 25s + 17 \cdot 2^{3k}$$

$$17(25s + 2^{3k})$$

So divisible 17

Since true for $n = 1$ and $n = k$ implies that true for $n = k+1$. therefore true for all $n \in \mathbb{Z}^+$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q7

It is given that $z = 5 + qi$ satisfies the equation $z^2 + iz = -p + 25i$, where $p, q \in \mathbb{R}$.

Find the value of p and the value of q.

Sol $(5+qi)^2 + i(5+qi) = -p + 25i$

$25 + q^2i^2 + 2\times5\times qi + 5i + qi^2 = -p + 25i$

$25 - q^2 + 10qi + 5i - q = -p + 25i$

compare real and imaginary part

$25 - q^2 - q = -p$ (Real)

$10q + 5 = 25$ (Imaginary)

$10q = 20 \therefore q = 2$

$25 - 4 - 2 = -p$

$19 = -p$

$p = -19$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q8

(a) Find $\int x(\ln x)^2 dx$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 dx = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

Sol (a) By parts

$u = (\ln x)^2 \quad v' = x$

$u' = \frac{2\ln x}{x} \quad v = \frac{x^2}{2}$

$\int x (\ln x)^2 dx$

$\int uv' = uv - \int u'v$

$= (\ln x)^2 \frac{x^2}{2} - \int \frac{\cancel{2} \cdot \ln x}{\cancel{x}} \times \frac{x^{\cancel{2}}}{\cancel{2}} dx$

$= (\ln x)^2 \frac{x^2}{2} - \int x \ln x \, dx$ Apply by parts again

$= (\ln x)^2 \frac{x^2}{2} - \ln x \cdot \frac{x^2}{2} + \int \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{2}$ $\quad u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}(\ln x) + \frac{x^2}{4} + c$

(b) $\int_1^4 x (\ln x)^2 dx = \left[\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}(\ln x) + \frac{x^2}{4}\right]_1^4$

$= \frac{16}{2}(\ln 4)^2 - \frac{16}{2}(\ln 4) + \frac{16}{4} - \frac{1}{2}(\ln 1)^2 + \frac{1}{2}(\ln 1) - \frac{1}{4}$

$= 8(\ln 4)^2 - 8(\ln 4) + 4 - \frac{1}{4}$

$= 8(2\ln 2)^2 - 8(2\ln 2) + \frac{15}{4}$

$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q9

Consider the function $f(x)=\dfrac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x\to 0} f(x)=16$, find the value of k. [6]

Sol (a)

$$f(-x) = f(x)$$

$$f(-x) = \frac{[\sin(-kx)]^2}{(-x)^2}$$

$$= \frac{[-\sin(kx)]^2}{(-x)^2}$$

$$= \frac{\sin^2 kx}{x^2}$$

hence $f(x)$ is even function

(b)

$$\lim_{x\to 0} f(x) = \frac{\sin^2 kx}{x^2}$$

$x \to 0$ $\quad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cancel{2}\sin kx \cdot \cos kx \cdot k}{\cancel{2}x} = k \lim_{x\to 0} \frac{\sin kx \cdot \cos kx}{x}$$

$x \to 0$ $\quad \frac{0}{0}$

$$k \left[\frac{\sin kx \cdot - \sin kx \cdot k + \cos kx \cdot k \cdot \cos x}{}\right]$$

$$= \frac{k^2[-\sin^2 kx + \cos^2 kx]}{1} = k^2 \lim_{x\to 0} \cos 2kx$$

$$= k^2 \quad \text{as } \cos 0 = 1$$

$\lim_{x\to 0} f(x) = 16 \quad \therefore \quad k^2 = 16$

$k = \pm 4$ hence $k = 4$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q10

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \dfrac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

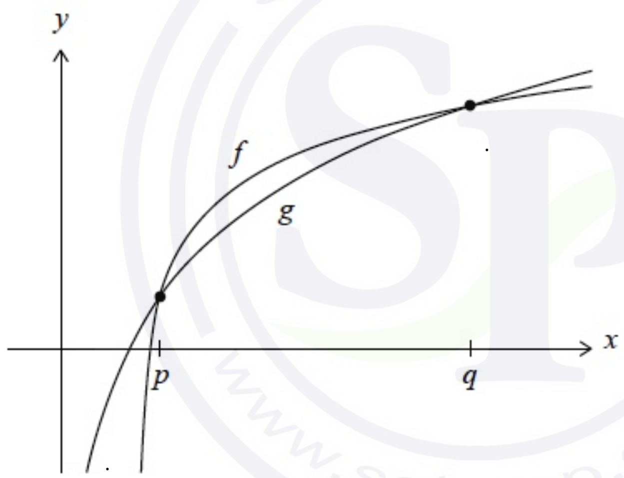
The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d. [9]

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$. [5]

Sol (a) $x > 0$

(b) (i) $\ln(2x-9) = 2\ln x - \ln d$

$\ln(2x-9) = \ln x^2 - \ln d$

$\ln(2x-9) = \ln \dfrac{x^2}{d}$

$d(2x-9) = x^2$

$x^2 - 2dx + 9d = 0$

b (ii) $b^2 - 4ac > 0$

Curves are intersecting at two distinct points

$(-2d)^2 - 4 \times 1 \times 9d > 0$

$4d^2 - 4 \times 9d > 0$

$d^2 - 9d > 0$

b (iii) $d(d-9) > 0$

$d < 0 \cup d > 9$

c $d = 10$

$x^2 - 20x + 90 = 0$

$x = \dfrac{+20 \pm \sqrt{400 - 360}}{2}$

$= \dfrac{20 \pm 2\sqrt{10}}{2} = 10 \pm \sqrt{10}$

$q = 10 + \sqrt{10}$ $p = 10 - \sqrt{10}$

$q - p = \cancel{10} + \sqrt{10} - \cancel{10} + \sqrt{10} = 2\sqrt{10}$

$\therefore$ $a = 2$ $b = 10$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q11

Consider the function $f(x) = e^{\cos 2x}$, where $-\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \le x \le \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4.

(ii) Hence, find the Maclaurin series for $e^{\cos 2x - 1}$, up to and including the term in x^4.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} e^{\cos 2x}\,dx \approx \frac{149e}{1500}$. [3]

Sol (a) $f'(x) = e^{\cos 2x} \cdot (-\sin 2x) \times 2$

$f'(x) = -2e^{\cos 2x} \sin 2x$

$f'(x) = 0$

$0 = -2e^{\cos 2x} \sin 2x$

$\sin 2x = 0 \qquad e^{\cos 2x} = 0 \ \times$

$2x = \sin^{-1} 0$

$x = 0, \quad x = \frac{\pi}{2}, \quad x = \pi$

$x = 0 \quad y = e^{\cos 0} = e$

$(0, e)$

$x = \frac{\pi}{2} \quad y = e^{\cos 2 \times \frac{\pi}{2}} = e^{-1}$

$\left(\frac{\pi}{2}, \frac{1}{e}\right)$

$x = \pi \quad y = e^{\cos 2\pi} = e^1$

(π, e)

(b) $f'(x) = -2e^{\cos 2x} \cdot \sin 2x$

$$f''(x) = -2\left[e^{\cos 2x} \cdot \cos 2x \cdot 2 + e^{\cos 2x} \cdot (-\sin 2x) \times 2 \times \sin 2x\right]$$

$$= -2\left[2e^{\cos 2x} \cdot \cos 2x - 2e^{\cos 2x}(\sin 2x)^2\right]$$

$$f''(x) = -4e^{\cos 2x} \cdot \cos 2x + 4e^{\cos 2x} \cdot (\sin 2x)^2$$

$x = 0$ $\quad f''(0) = -4e^{\cos 0}\cos(0) + 4e^{\cos 0}(\sin 0)^2$

$$= -4e < 0$$

So maximum at $(0, e)$ ✓

$x = \frac{\pi}{2}$ $\quad f''(\frac{\pi}{2}) = -4e^{\cos \pi}\cos(\pi) + 4e^{\cos \pi}(\sin \pi)^2$

$$= -4e^{-1}(-1) + 0$$

$$= \frac{4}{e} > 0$$

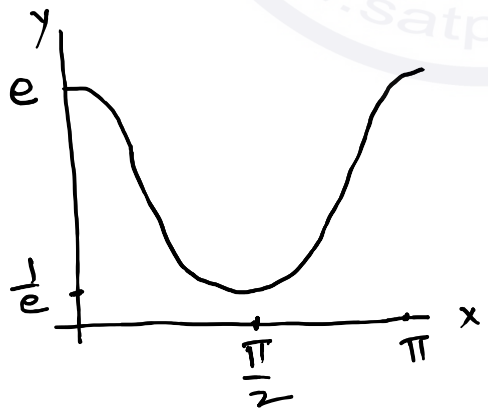
So minimum at $(\frac{\pi}{2}, \frac{1}{e})$ ✓

$x = \pi$ $\quad f''(\pi) = -4e^{\cos 2\pi}\cos 2\pi + 4e^{\cos 2\pi}(\sin 2\pi)^2$

$$= -4e < 0$$

So maximum at (π, e) ✓

(c)



(d) (i) Maclurin Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$f(x) = \cos 2x$$

$$f(0) = \cos(0) = 1$$

$$f'(0) = -2\sin(2\times 0) = 0$$

$$f''(0) = -4\cos 2x = -4\cos 0 = -4$$

$$f'''(0) = -4(-\sin 2x)\times 2 = 8\sin(0) = 0$$

$$f^{iv}(0) = 8\cos 2x \times 2 = 16\cos(0) = 16$$

$$f(x) = 1 + 0 - 4\frac{x^2}{2!} + 0 + \frac{16x^4}{4!}$$

$$= 1 - 2x^2 + \frac{2}{3}x^4$$

(ii) $$e^{\cos 2x - 1} = e^{\cancel{1} - 2x^2 + \frac{2}{3}x^4 - \cancel{1}}$$

$$= e^{-2x^2 + \frac{2}{3}x^4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-2x^2 + \frac{2}{3}x^4} = 1 + \left(-2x^2 + \frac{2}{3}x^4\right) + \frac{\left(-2x^2 + \frac{2}{3}x^4\right)^2}{2!} + \cdots$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + 2x^4$$

$$= 1 - 2x^2 + \frac{8}{3}x^4$$

(iii) $$f(x) \approx e\left[1 - 2x^2 + \frac{8}{3}x^4\right]$$

(e) $$e\int_0^{\frac{1}{10}} (1 - 2x^2)\,dx = e\left[x - \frac{2x^3}{3}\right]_0^{\frac{1}{10}}$$

$$= e\left[\frac{1}{10} - \frac{2}{3000}\right]$$

$$= e\left[\frac{300 - 2}{3000}\right]$$

$$= e\,\frac{149}{1500}$$

Problem - N23/5/MATHX/HP1/ENG/TZ1/XX/Q12

(a) Find the binomial expansion of $(\cos\theta + i\sin\theta)^5$. Give your answer in the form $a + bi$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

Sol (a) $(\cos\theta + i\sin\theta)^5 =$

$\cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$

$\cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

(b) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

equate imaginary part

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin 5\theta = 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1 + \sin^4\theta - 2\sin^2\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 5\sin\theta + 5\sin^5\theta - 10\sin^3\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

c (i) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$= \sin\theta\,(16\sin^4\theta - 20\sin^2\theta + 5)$

$\theta = \frac{\pi}{5}$ $\quad \sin 5\times\frac{\pi}{5} = \sin\pi = 0$

$\theta = \frac{3\pi}{5}$ $\quad \sin 5\times\frac{3\pi}{5} = \sin 3\pi = 0$

$0 = \sin\theta\,(16\sin^4\theta - 20\sin^2\theta + 5)$

$\sin\theta = 0$ $\quad 16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\theta = \frac{\pi}{5}$ $\quad \sin\frac{\pi}{5} \neq 0$ $\qquad \theta = \frac{3\pi}{5}$ $\quad \sin\frac{3\pi}{5} \neq 0$

therefore

$\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solution of

$16\sin^4\theta - 20\sin^2\theta + 5$

c (ii) $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$$\sin^2\theta = \frac{-(-20) \pm \sqrt{400 - 320}}{32}$$

$$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$$

$$\sin\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

$$\sin\frac{\pi}{5}.\sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}} \times \sqrt{\frac{5-\sqrt{5}}{8}}$$

$$= \sqrt{\frac{20}{64}}$$

$$= \frac{\sqrt{5}}{4}$$