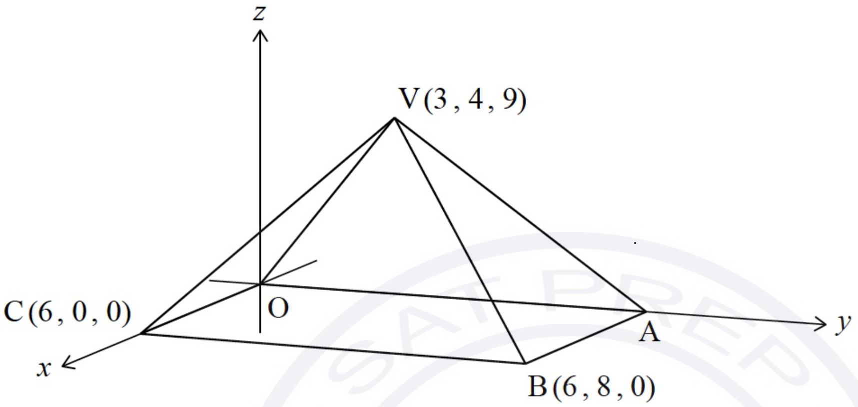
Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q1

The following diagram shows a pyramid with vertex V and rectangular base OABC.

Point B has coordinates $(6, 8, 0)$, point C has coordinates $(6, 0, 0)$ and point V has coordinates $(3, 4, 9)$.

diagram not to scale



(a) Find BV. [2]

(b) Find the size of $\mathrm{B\hat{V}C}$. [4]

<u>Sol</u> (a) $BV = \sqrt{(6-3)^2 + (8-4)^2 + (0-9)^2}$

$= 10.29563$

$= 10.3$

(b) $CV = \sqrt{(6-3)^2 + (0-4)^2 + (0-9)^2}$

$= 10.29563$

$= 10.3$

$CB = \sqrt{(6-6)^2 + (0-8)^2 + (0-0)^2}$

$= 8$

$\angle BVC = \cos^{-1}\left[\dfrac{BV^2 + CV^2 - BC}{2 \times BV \times CV}\right]$

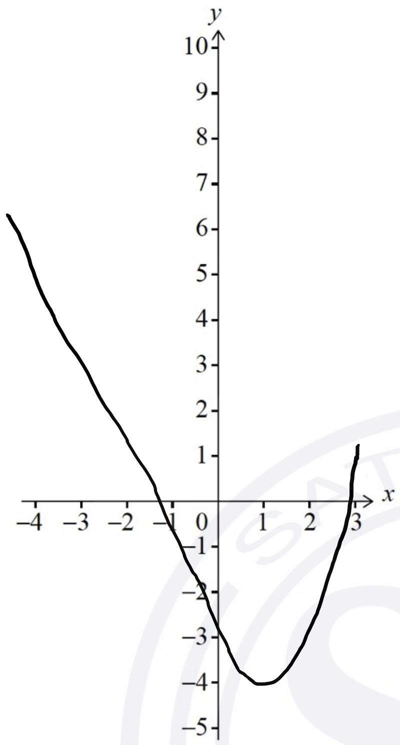
$= \cos^{-1}\left[\dfrac{10.3^2 + 10.3^2 - 8^2}{2 \times 10.3 \times 10.3}\right]$

$= 45.7^\circ$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q2

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \le x \le 3$. [3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k, followed by a vertical translation of c units.

Find the value of k and the value of c. [2]

Sol (a) On the axes.

(b) $e^x - 3x - 4$

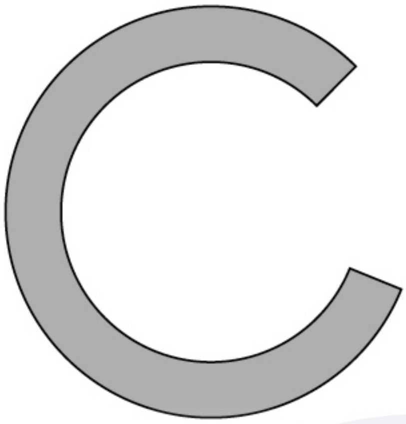
$e^{2x} - 6x - 7$

horizontal stretch by scale factor $k = \frac{1}{2}$

and vertical translation $c = -3$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q3

A company is designing a new logo in the shape of a letter "C".



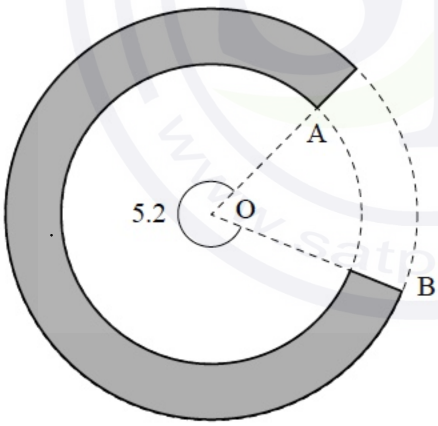
The letter "C" is formed between two circles with centre O.

The point A lies on the circumference of the inner circle with radius r cm, where $r < 10$.

The point B lies on the circumference of the outer circle with radius 10 cm.

The reflex angle $\text{A}\hat{\text{O}}\text{B}$ is 5.2 radians. The letter "C" is shown by the shaded area in the following diagram.

diagram not to scale



(a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is 64 cm^2.

(b) (i) Find the value of r.

(ii) Find the perimeter of the "C". [5]

Sol (a) Area of the Outer Sector

$= \frac{1}{2} \times 10^2 \times 5.2 = 260$

Area of the inner sector

$= \frac{1}{2} \times r^2 \times 5.2 = 2.6r^2$

Area of the C

= Area of the outer Sector − Area of the inner sector

$= 260 - 2.6r^2$

(b) (i) $64 = 260 - 2.6r^2$

$2.6r^2 = 260 - 64$

$r = \sqrt{\frac{260-64}{2.6}}$

$= 8.682431$

$= 8.68$ cm

(b) (ii) Perimeter of the C

= Arc length of the outer sector + Arc length of the inner Sector + Difference of radius × 2

$= 10 \times 5.2 + 8.682431 \times 5.2 + (10 - 8.682431) \times 2$

$= 99.78378$

$= 99.8$ cm

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q4

A particle moves along a straight line. Its displacement, s metres, from a fixed point O after time t seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

The particle first comes to rest after q seconds.

(a) Find the value of q. [2]

(b) Find the total distance that the particle travels in the first q seconds. [3]

Sol (a) $s(t) = 4.3\sin(\sqrt{3t+5})$

$0 \le t \le 10$

$q = 5.7355$

≈ 5.74

(b) $\int_0^{5.74} |v(t)|\,dt$

$= 7.68$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q5

The following table shows the probability distribution of a discrete random variable X, where $a, k \in \mathbb{R}^+$.

x	1	2	3	4
$P(X=x)$	k	k^2	a	k^3

Given that $E(X) = 2.3$, find the value of a.

Sol.

$$k + k^2 + a + k^3 = 1 \quad \text{—— (i)}$$

$$1 \times k + 2 \times k^2 + 3 \times a + 4 \times k^3 = 2.3$$

$$k + 2k^2 + 3a + 4k^3 = 2.3 \quad \text{—— (ii)}$$

Substitute value of a from eq(i) to eq.(2)

$$a = 1 - k - k^2 - k^3$$

$$k + 2k^2 + 3(1 - k - k^2 - k^3) + 4k^3 = 2.3$$

$$k^3 - k^2 - 2k + 0.7 = 0$$

$$k = 0.3158707$$

$$a = 1 - (0.3158707) - (0.3158707)^2 - (0.3158707)^3$$

$$= 0.5528392$$

$$= 0.553$$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q6

The random variable X is such that $X \sim B(25, p)$ and $\text{Var}(X) = 5.75$.

(a) Find the possible values of p. [3]

The random variable Y is such that $Y = 1 - 2X$.

(b) Find $\text{Var}(Y)$. [2]

Sol (a)

$$\text{Var}(x) = npq$$
$$= 25p(1-p)$$
$$5.75 = 25p - 25p^2$$
$$25p^2 - 25p + 5.75 = 0$$
$$p = 0.6414 \quad p = 0.358579$$
$$= 0.641 \quad p = 0.358$$

(b)

$$Y = 1 - 2X$$
$$\text{Var}(Y) = \text{Var}(1) + 4\,\text{Var}(x)$$
$$= 0 + 4 \times 5.75$$
$$= 23$$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q7

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

(a) In how many ways can the team members be placed if

(i) there are no restrictions;

(ii) the girls must be placed next to each other. [3]

(b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]

Sol a(i) $9! = 362880$

(ii) $3!\,7! = 30240$

(b) $\binom{6}{3} \times \binom{3}{2} + \binom{6}{2}\binom{3}{3}$

$20 \times 3 + 15 \times 1$

$= 75$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q8

Three points are given by $\mathrm{A}(0, p, 2)$, $\mathrm{B}(1, 1, 1)$ and $\mathrm{C}(p, 0, 4)$, where p is a positive constant.

(a) Show that $\vec{\mathrm{AB}} \times \vec{\mathrm{AC}} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

(b) Hence, find the smallest possible value of $\left|\vec{\mathrm{AB}} \times \vec{\mathrm{AC}}\right|^2$. [3]

(c) Hence, find the smallest possible area of triangle ABC. [2]

Sol (a)

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ p \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-p \\ -1 \end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= \begin{pmatrix} p \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ p \\ 2 \end{pmatrix} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1-p & -1 \\ p & -p & 2 \end{vmatrix}$$

$$= i(2(1-p) - p) - j(2+p) + k(-p - p(1-p))$$

$$= i(2-3p) - j(2+p) + k(-2p + p^2)$$

$$= \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$$

(b) $$|\vec{AB} \times \vec{AC}|^2 = (2-3p)^2 + (-2-p)^2 + (p^2-2p)^2$$

$$= 6.75$$

(c) Area of triangle

$$= \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \times \sqrt{6.75} = 1.299$$

$$\approx 1.30$$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q9

Consider the differential equation $\frac{dy}{dx} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2e^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

Sol (a) $x_{n+1} = x_n + \text{step size}$ $\quad y_{n+1} = y_n + \text{step size} \times f(x_n y_n)$

$x_0 = 0 \quad y_0 = 2$

$x_1 = 0 + 0.1 = 0.1 \qquad y_1 = 2 + 0.1 \times \frac{4-2}{10} = 2.02$

$x_2 = 0.1 + 0.1 = 0.2 \qquad y_2 = 2.02 + 0.1 \times \frac{4-2.02}{10} = 2.0398$

$x_3 = 0.2 + 0.1 = 0.3 \qquad y_3 = 2.0398 + 0.1 \times \frac{4-2.0398}{10} = 2.0594$

$x_4 = 0.3 + 0.1 = 0.4 \qquad y_4 = 2.0594 + 0.1 \times \frac{4-2.0594}{10}$

$= 2.07881$

$x_5 = 0.4 + 0.1 = 0.5 \qquad y_5 = 2.07881 + 0.1 \times \frac{4-2.07881}{10}$

$= 2.09802$

(b) $\frac{dy}{dx} = \frac{4-y}{10}$

by variable seperable

$\int \frac{dy}{4-y} = \int \frac{dx}{10}$

$\frac{\ln|4-y|}{-1} = \frac{x}{10} + c$

$x = 0 \quad y = 2$

$\frac{\ln|4-2|}{-1} = c$

$c = -\ln 2$

$$\frac{\ln|4-y|}{-1} = \frac{x}{10} - \ln 2$$

$$-\ln|4-y| + \ln 2 = \frac{x}{10}$$

$$\frac{2}{4-y} = e^{\frac{x}{10}}$$

$$4-y = 2e^{-\frac{x}{10}}$$

$$y = 4 - 2e^{-x/10}$$

(c) $x = 0.5 \quad y = 4 - 2e^{-\frac{0.5}{10}}$

Absolute difference

$= |2.0980199 - 2.09754115|$

$= 0.0004787$

$= 4.79 \times 10^{-4}$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q10

A farmer is growing a field of wheat plants. The height, $H\,\text{cm}$, of each plant can be modelled by a normal distribution with mean μ and standard deviation σ.

It is known that $P(H < 94.6) = 0.288$ and $P(H > 98.1) = 0.434$.

(a) Find the probability that the height of a randomly selected plant is between $94.6\,\text{cm}$ and $98.1\,\text{cm}$. [2]

(b) Find the value of μ and the value of σ. [5]

The farmer measures 100 randomly selected plants. Any plant with a height greater than $98.1\,\text{cm}$ is considered ready to harvest. Heights of plants are independent of each other.

(c) (i) Find the probability that exactly 34 plants are ready to harvest.

(ii) Given that fewer than 49 plants are ready to harvest, find the probability that exactly 34 plants are ready to harvest. [6]

In another field, the farmer is growing the same variety of wheat, but is using a different fertilizer. The heights of these plants, $F\,\text{cm}$, are normally distributed with mean 98.6 and standard deviation d. The farmer finds the interquartile range to be $4.82\,\text{cm}$.

(d) Find the value of d. [3]

Sol (a) $P(94.6 < H < 98.1)$

$= P(H < 98.1) - P(H < 94.6)$

$= 1 - 0.434 - 0.288$

$= 0.278$

(b) $P(H < 98.1) = 0.566$

$$\frac{98.1 - \mu}{\sigma} = \text{InvN}(0.566)$$

$98.1 = \mu + \text{InvN}(0.566)\sigma$

$P(H < 94.6) = 0.288$

$94.6 = \mu + \text{InvN}(0.288)\sigma$

$98.1 = \mu + 0.1661\sigma$

$94.6 = \mu - 0.5592\sigma$

$\mu = 97.3 \quad \sigma = 4.82$

(c) (i) $H \sim B(100, 0.434)$

$P(H = 34)$

$= \binom{100}{34}(0.434)^{34}(1-0.434)^{100-34}$

$= 0.01332$

(ii) $P(H = 34 \mid H < 49)$

$= \dfrac{P(H=34)}{P(H<49)}$

$= \dfrac{0.01332}{0.8482}$

$= 0.0157$

(d) $F \sim N(98.6, d^2)$

We know IQR $= 4.82$

$Q_3 = 98.6 + \dfrac{4.82}{2} = 101.01$

$Q_1 = 98.6 - \dfrac{4.82}{2} = 96.19$

So $P(96.19 < F < 101.01) = .75 - .25$

$= .5$

$P(F < 101.01) - P(F < 96.19) = 0.5$

$P\left(Z < \dfrac{101.01 - 98.6}{d}\right) - P\left(Z < \dfrac{96.19 - 98.6}{d}\right) = 0.5$

$P\left(Z < \dfrac{2.41}{d}\right) - P\left(Z < -\dfrac{2.41}{d}\right) = 0.5$

$1 - P\left(Z < -\dfrac{2.41}{d}\right) - P\left(Z < -\dfrac{2.41}{d}\right) = 0.5$

$$1-2P\left(Z<-\frac{2.41}{d}\right)=0.5$$

$$P\left(Z<-\frac{2.41}{d}\right)=\frac{1-0.5}{2}=0.25$$

$$-\frac{2.41}{d}=\text{InvN}(0.25)$$

$$-\frac{2.41}{d}=-0.67449$$

$$d=3.57307$$

$$=3.57$$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q11

Consider the function defined by $f(x)=\dfrac{x^2-14x+24}{2x+6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f. [1]

(b) Find the coordinates of the points where the graph of f crosses the x-axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f. [4]

(f) Solve the inequality $f(x) > x$. [4]

Sol (a) $2x+6=0$

$x=-3$

(b) When graph f cross x-axis mean we have find x-intercepts

$x=2 \quad x=12$

So $(2,0)$ and $(12,0)$

(c)

$$2x+6 \overline{)\, x^2-14x+24\,}\left(\tfrac{1}{2}x-\tfrac{17}{2}\right.$$

$$\underline{\cancel{x^2} + 3x}$$

$$-17x+24$$

$$\underline{-17x-51}$$

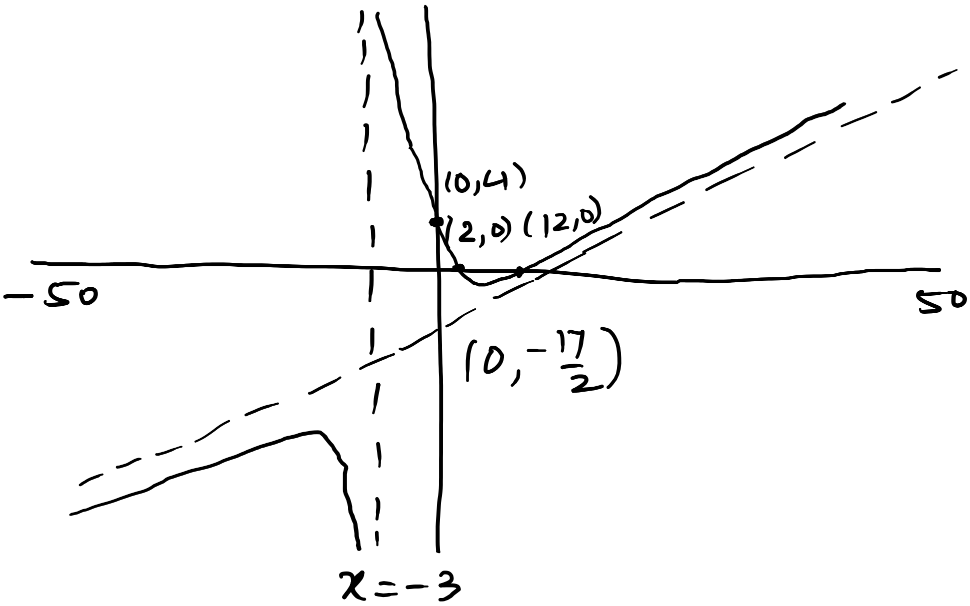
$$75$$

So equation oblique asymptote

$$y=\frac{1}{2}x-\frac{17}{2}$$

$$a=\frac{1}{2} \quad b=-\frac{17}{2}$$

(d)



(e) Range

Minimum point Coordinate

$= (5.66025, -1.33975)$

Maximum point coordinate

$= -11.6603, -18.6603$

So range would be

$f(x) \leq -18.7$ and $f(x) \geq -1.34$

(f) Solve $f(x) > x$

So point of intersection

$(1.13553, 1.13553)$

$(-21.1355, -21.1355)$

$x < -21.1$ and $-3 < x < 1.14$

Problem - N23/5/MATHX/HP2/ENG/TZ0/XX/Q12

Line L is given by the vector equation $r_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line M is given by the vector equation $r_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines L and M intersect at a point A and find the position vector of A. [5]

(b) Verify that the lines L and M both lie in the plane Π given by $r \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point B has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through B perpendicular to Π intersects Π at point C.

(c) (i) Find the position vector of C.

(ii) Hence, find $\left|\overrightarrow{\mathrm{BC}}\right|$. [7]

(d) Find the reflection of the point B in the plane Π. [3]

Sol (a)

$$1 + 2s = 9 + 4t \checkmark$$
$$2 + 3s = 9 + t \checkmark$$
$$-3 + 6s = 11 + 2t$$

$$2s - 4t = 8 \quad \text{—(i)}$$
$$3s - t = 7 \quad \text{—(ii)}$$

$$s = 2 \qquad t = -1$$

$$\begin{pmatrix} 1 + 2\times 2 \\ 2 + 3\times 2 \\ -3 + 6\times 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 9 + (-1)4 \\ 9 + (-1)1 \\ 11 + (-1)2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$$

So position of A $= \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 5\times0+8\times2+9\times-1$

$= 16-9=7$

hence Lines L and M lie in given plane Π

(c) So equation of line passing through B and perpendicular to plane Π

$$r = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

point on line

$$\begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix}$$

As line intersect plane Π at C

therefore

$$\begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$$

$$24+4\lambda-2+\lambda=7$$

$$5\lambda = 7-22$$

$$5\lambda = -15$$

$$\lambda = -3$$

(i) Position vector C $= \begin{matrix} -3 \\ 12+2(-3) \\ 2-(-3 \end{matrix}$

$$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

(ii) $\vec{BC} = \vec{OC} - \vec{OB}$

$$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$$

$$|\vec{BC}| = \sqrt{0 + (-6)^2 + 3^2}$$

$$= \sqrt{45} = 3\sqrt{5}$$

(d) Let B' be the image of B

$$\vec{OB'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

We know

$$\mu = 2\lambda = -6$$

$$OB' = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} - 6 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$$