Subject – Math AA(Higher Level)
Topic - Statistics and Probability
Year - May 2021 – Nov 2024
Paper -3
Questions

Question 1

[Maximum mark: 31]

This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of x-intercepts.

In parts (a) – (f), consider quadratic functions, $f(x) = ax^2 + bx + c$, whose coefficients, a, b and c, are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a 2, 3 and 5 in turn generates the quadratic function $f(x) = 2x^2 + 3x + 5$.

(a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

(b) The set of coefficients, $a = 1$, $b = 4$ and $c = 4$, is randomly generated to form the quadratic function $f(x) = x^2 + 4x + 4$.

Verify that this graph of f has only one x-intercept. [2]

(c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one x-intercept is $\frac{5}{216}$. [6]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct** x-intercepts.

(d) By considering the discriminant, determine the set of possible values of ac. [3]

(e) (i) For the case where $ac = 1$, show that there are four quadratic functions whose corresponding graphs have two distinct x-intercepts. [1]

(ii) For the case where $ac = 2$, show that there are eight quadratic functions whose corresponding graphs have two distinct x-intercepts. [2]

Let p be the probability of the graph of such a randomly generated quadratic function having two distinct x-intercepts.

(f) Using the approach started in part (e), or otherwise, find the value of p. [6]

In parts (g) and (h), consider a randomly generated quadratic function, $f(x) = x^2 + 2Zx + 1$, where the continuous random variable $Z \sim \mathrm{N}(0, 1)$.

(g) Find the probability that the graph of f has two x-intercepts. [3]

The continuous random variables, X_1 and X_2, represent the x-intercepts of the graph of f where $X_1 = -Z - \sqrt{Z^2 - 1}$ and $X_2 = -Z + \sqrt{Z^2 - 1}$.

(h) Given that the graph of f has two x-intercepts, X_1 and X_2, find the probability that both X_1 and X_2 are greater than 0.5. [7]

Question 2

[Maximum mark: 27]

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(i) Find the probability that the first boost occurs on the third action. [2]

(ii) Find the probability that at least one boost occurs in the first six actions. [3]

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1-p)^{x-1}$. [1]

Let X be the number of actions until the first boost occurs.

(ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

Consider the sum of an infinite geometric sequence, with first term a and common ratio $r\ (|r| < 1)$,

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

(c) (i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

(ii) Hence, show that $\mathrm{E}(X) = \dfrac{1}{p}$. [2]

It can be shown that $\text{Var}(X) = \dfrac{1-p}{p^2}$.

(d) Find $\text{E}(X)$ and $\text{Var}(X)$ when $p = 0.1$. [2]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$\text{P}(Y = y)$	0.2	m	0.288	n	0.0384

(g) (i) Find the value of m and the value of n. [2]

(ii) Show that $\text{E}(Y) = 2.5104$. [2]

(iii) Find $\text{Var}(Y)$. [2]

(h) (i) Use the expression given in (c)(ii) to find the value of p for which $\text{E}(X) = \text{E}(Y)$. [1]

(ii) Find $\text{Var}(X)$ for this value of p. [1]

(iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]