Subject – Math AI(Higher Level)
Topic - Functions
Year - May 2021 – Nov 2024
Paper -3
Questions

Question 1

[Maximum mark: 28]

This question is about modelling the spread of a computer virus to predict the number of computers in a city which will be infected by the virus.

A systems analyst defines the following variables in a model:

- t is the number of days since the first computer was infected by the virus.
- $Q(t)$ is the total number of computers that have been infected up to and including day t.

The following data were collected:

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146

(a) (i) Find the equation of the regression line of $Q(t)$ on t. [2]

(ii) Write down the value of r, Pearson's product-moment correlation coefficient. [1]

(iii) Explain why it would not be appropriate to conduct a hypothesis test on the value of r found in (a)(ii). [1]

A model for the early stage of the spread of the computer virus suggests that

$$Q'(t) = \beta N Q(t)$$

where N is the total number of computers in a city and β is a measure of how easily the virus is spreading between computers. Both N and β are assumed to be constant.

(b) (i) Find the general solution of the differential equation $Q'(t) = \beta N Q(t)$. [4]

(ii) Using the data in the table write down the equation for an appropriate non-linear regression model. [2]

(iii) Write down the value of R^2 for this model. [1]

(iv) Hence comment on the suitability of the model from (b)(ii) in comparison with the linear model found in part (a). [2]

(v) By considering large values of t write down one criticism of the model found in (b)(ii). [1]

(c) Use your answer from part (b)(ii) to estimate the time taken for the number of infected computers to double. [2]

The data above are taken from city X which is estimated to have 2.6 million computers. The analyst looks at data for another city, Y. These data indicate a value of $\beta = 9.64 \times 10^{-8}$.

(d) Find in which city, X or Y, the computer virus is spreading more easily. Justify your answer using your results from part (b). [3]

An estimate for $Q'(t)$, $t \geq 5$, can be found by using the formula:

$$Q'(t) \approx \frac{Q(t+5) - Q(t-5)}{10}.$$

The following table shows estimates of $Q'(t)$ for city X at different values of t.

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146
$Q'(t)$		a	171.6	766.7	b	11 207.6	

(e) Determine the value of a and of b. Give your answers correct to one decimal place. [2]

An improved model for $Q(t)$, which is valid for large values of t, is the logistic differential equation

$$Q'(t) = kQ(t)\left(1 - \frac{Q(t)}{L}\right)$$

where k and L are constants.

Based on this differential equation, the graph of $\frac{Q'(t)}{Q(t)}$ against $Q(t)$ is predicted to be a straight line.

(f) (i) Use linear regression to estimate the value of k and of L. [5]

(ii) The solution to the differential equation is given by

$$Q(t) = \frac{L}{1 + Ce^{-kt}}$$

where C is a constant.

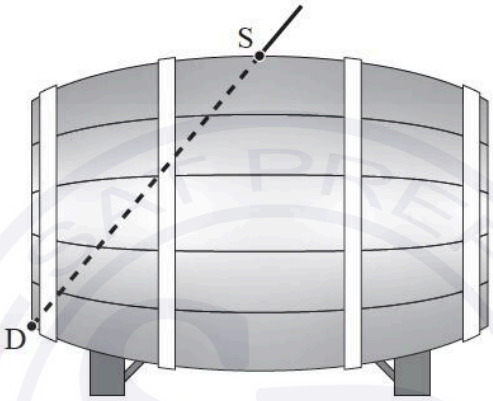
Using your answer to part (f)(i), estimate the percentage of computers in city X that are expected to have been infected by the virus over a long period of time. [2]

Question 2

[Maximum mark: 26]

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD. The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

(a) Given that C was directly proportional to d, find an equation for C in terms of d. [3]

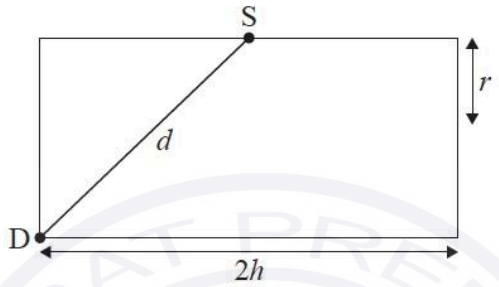
A particular barrel of wine cost 0.96 gulden.

(b) Show that $d = 0.6$. [1]

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD.

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

diagram not to scale



(c) Find an expression for r^2 in terms of d and h. [3]

Let the volume of this barrel be $V\,\text{m}^3$.

(d) Show that $V = \frac{\pi}{2}\left(d^2h - h^3\right)$. [2]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

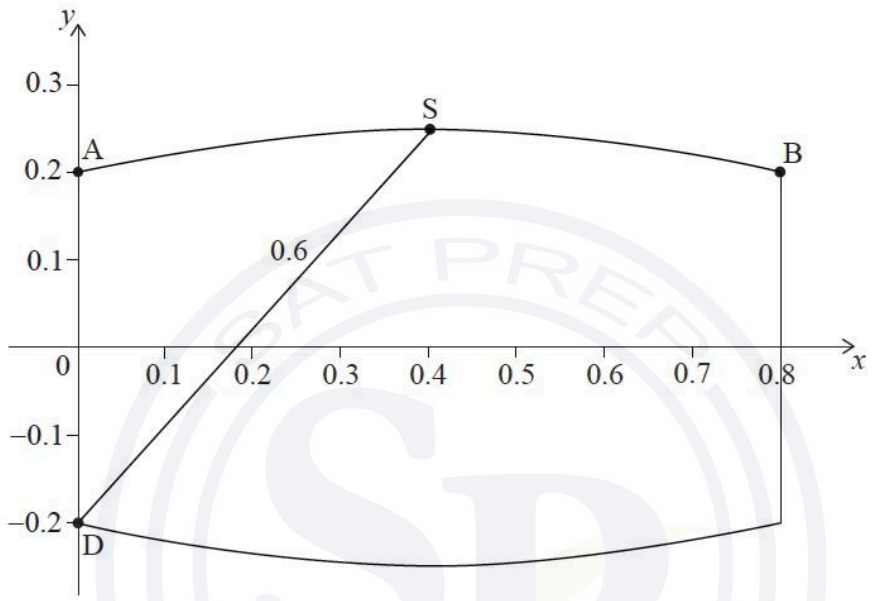
(e) (i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]

(ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$. [3]

(iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius $0.2\,\text{m}$ and whose length is $0.8\,\text{m}$.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c$, $0 \le x \le 0.8$, about the x-axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $\text{SD} = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

The coordinates of A and B are $(0,\ 0.2)$ and $(0.8,\ 0.2)$ respectively.

(f) Find the equation of the quadratic curve, ASB. [6]

(g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]

(h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]