2225 – 7106M

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 1

25 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. $f'(x) = 4x^2 - 16$ ***A1***

sets their derivative equal to zero ***(M1)***

$4x^2 - 16 = 0$, ($x = \pm 2$)

$p = 2$ (accept $x = 2$) ***A1***

substitutes their **positive** *p* into $f(x)$ ***(M1)***

$$y = \frac{4\left(2^3\right)}{3} - 16(2)\left(= \frac{32}{3} - 32 = -\frac{64}{3}\right)$$

$q = -\frac{64}{3}$ (accept $y = \frac{-64}{3}$) ***A1***

Total [5 marks]

2. (a) $k=\frac{4}{400}\left(=\frac{1}{100}=0.01\right)$ ***A1***

[1 mark]

(b) attempt to find binomial coefficients or multiply out brackets ***(M1)***

e.g. Pascal's triangle down to correct row OR $\left(1+2x+x^2\right)^2$ OR substitute into binomial expansion

$(1+x)^4$

$=1+4x+6x^2+4x^3+x^4$ ***A1***

[2 marks]

(c) **METHOD 1**

recognition that the expansion can be used with x replaced with k ***(M1)***

$\left(1+\frac{1}{100}\right)^4$

$=1+\frac{4}{100}+\frac{6}{100^2}+\ldots(=1+0.04+0.0006+\ldots)$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000\left(1+\frac{1}{100}\right)^4$

$=1000+40+0.6+\ldots(=1040.6\ldots)$

$=1041$ (dinar) ***A1***

METHOD 2

attempt to find the value of $(1+k)^4$ by hand ***(M1)***

$(1.01)^4=(1.0201)(1.01)^2=(1.030301)(1.01)$

$=1.0406\ldots$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000(1.01)^4$

$=1040.6\ldots$

$=1041$ (dinar) ***A1***

[4 marks]

Total [7 marks]

3. METHOD 1

attempt to set up integral $e^x - \left(-e^x\right) = 2e^x$ or e^x and then double ***(M1)***

$\int \left(e^x - \left(-e^x\right)\right) dx$ OR $2\int e^x\, dx$

$= 2\int_{-1}^{1} e^x\, dx$

$= 2\left[e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into their integrated function and subtract ***(M1)***

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

METHOD 2

$\int_{-1}^{1} e^x\, dx = \left[e^x\right]_{-1}^{1}$ and $\int_{-1}^{1} -e^x\, dx = \left[-e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into both their integrated functions and subtract ***(M1)***

$e^1 - e^{-1}$ and $-e^1 - (-e^{-1})$

subtracts their two integrals in correct order ***(M1)***

$e^1 - e^{-1} - \left(-e^1 + e^{-1}\right)$

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

Total [4 marks]

4. (a) $\mathrm{P}(A) = \dfrac{1}{4}$ ***(A1)***

attempt to use $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$ ***(M1)***

$$\frac{2}{3} = \frac{\mathrm{P}(B \cap A)}{\left(\frac{1}{4}\right)}$$

$$\mathrm{P}(A \cap B) = \frac{2}{3}\left(\frac{1}{4}\right)$$

$$= \frac{2}{12}\left(= \frac{1}{6}\right)$$ ***A1***

[3 marks]

(b) attempt to use $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ OR a Venn diagram, with their values of $\mathrm{P}(A)$ and $\mathrm{P}(B \cap A)$ ***M1***

$$\frac{3}{4} = \frac{1}{4} + \mathrm{P}(B) - \frac{1}{6}$$

$$\mathrm{P}(B) = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{4}{6}\left(= \frac{2}{3}\right)$$ ***A1***

$\mathrm{P}(B|A) = \mathrm{P}(B)$ OR $\mathrm{P}(A)\mathrm{P}(B) = \dfrac{1}{6}$ so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ (hence A and B are independent) ***R1***

Note: The ***R1*** is dependent on all previous marks

[3 marks]

Total [6 marks]

5. (a) (i) for a sequence of areas, uses two consecutive terms to find a common ratio OR for sequences of both widths and heights uses two consecutive terms for both sequences to find both common ratios OR recognises that both widths and heights are geometric sequences with common ratio $\frac{3}{2}$ ***M1***

areas form a geometric sequence with first term 20 and common ratio $\frac{45}{20}$ ***A1***

OR area of picture frame F_n is $4\left(\frac{3}{2}\right)^{n-1} \times 5\left(\frac{3}{2}\right)^{n-1}$

area of F_n is $20\left(\frac{9}{4}\right)^{n-1}$ ***AG***

(ii) attempt to find the sum of the areas using $S_n = \frac{u_1\left(r^n - 1\right)}{r-1}$ ***(M1)***

sum of areas $\frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1}\left(= 16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***(A1)***

mean area $= \frac{1}{10}\left(\frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1}\right)\left(= \frac{1}{10}\left(16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)\right)$

$= \frac{16}{10}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\left(= \frac{8}{5}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***A1***

$p = \frac{8}{5},\ a = 10$

[5 marks]

continued…

Question 5 continued.

(b) recognition that median is between 5th and 6th picture frame ***(M1)***

median area $= \dfrac{20\left(\dfrac{9}{4}\right)^4 + 20\left(\dfrac{9}{4}\right)^5}{2}$ ***(A1)***

$= \dfrac{20\left(\dfrac{9}{4}\right)^4\left(1+\dfrac{9}{4}\right)}{2}$

$= \dfrac{65}{2}\left(\dfrac{9}{4}\right)^4$ ***A1***

$q = \dfrac{65}{2}$

[3 marks]

Total [8 marks]

6. direction vectors are $a\boldsymbol{j}+\boldsymbol{k}\left(=\begin{pmatrix}0\\a\\1\end{pmatrix}\right)$ and $\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\left(=\begin{pmatrix}1\\2\\3\end{pmatrix}\right)$ ***(A1)***

recognition that the scalar product of the direction vectors is 0 ***(M1)***

$\begin{pmatrix}0\\a\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\3\end{pmatrix}(=2a+3)=0$

$a=-\dfrac{3}{2}$ ***A1***

at point of intersection $4=1+\mu$, $-\dfrac{3}{2}\lambda=2\mu$ and $-1+\lambda=-b+3\mu$ ***(A1)***

attempt to solve 3 equations in μ, λ and b, derived from the point of intersection, to find μ, λ and b ***(M1)***

$\mu=3,\ \lambda=-4$

$b=14$ ***A1***

Total [6 marks]

7. (a) $(3=)e^{\ln 3}$ OR $a=\ln 3$ ***A1***

[1 mark]

(b) (i) $z=\frac{1}{3}e^{\mathrm{i}\ln 3}$ OR $\left(\mathrm{Re}(z)\right)=e^{-\ln 3}\cos\left(\ln 3\right)$ ***(A1)***

$\left(\mathrm{Re}(z)\right)=\frac{1}{3}\cos\left(\ln 3\right)$ ***A1***

(ii) $\frac{1}{z}=3e^{-\mathrm{i}\ln 3}\left(=3\left(\cos\left(-\ln 3\right)+\mathrm{i}\sin\left(-\ln 3\right)\right)\right)$ OR $\left(\mathrm{Re}\left(\frac{1}{z}\right)=\right)e^{\ln 3}\cos\left(-\ln 3\right)$ ***(A1)***

$\mathrm{Re}\left(\frac{1}{z}\right)=3\cos\left(-\ln 3\right)$

$=3\cos(\ln 3)$ OR $3\cos\left(\ln\frac{1}{3}\right)$ ***A1***

[4 marks]

Total [5 marks]

8. (a) $1+\log_2 n = \log_2 2 + \log_2 n$

$= \log_2(2n)$ ***A1***

$2n \ge n+1$ OR $\log_2(2n) \ge \log_2(n+1)$ ***R1***

(for $n \in \mathbb{Z}^+$) (since $\log_2 x$ is an increasing function)

$1+\log_2 n \ge \log_2(n+1)$ (for $n \in \mathbb{Z}^+$) ***AG***

Note: Do not award ***A0R1***.

[2 marks]

(b) for $n=1$

$\log_2 1 = 0$ and $1>0$ OR LHS $=1$, RHS = $\log_2 1 = 0$ OR $\log_2 2 > \log_2 1$ ***R1***

(so true for $n=1$)

assume true for $n=k$, ie $k > \log_2 k$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$", "assume $n=k$ is true". The assumption of truth must be clear. "Assume P_k true" is accepted.

The following two marks after this ***M1*** are independent of this mark and can be awarded.

hence

$1+k > 1+\log_2 k$ (using assumption) ***M1***

$\ge \log_2(k+1)$ (using result from part a)) ***A1***

hence if true for $n=k$ then true for $n=k+1$ ***R1***

and as true for $n=1$, therefore true for all $n \in \mathbb{Z}^+$.

Note: Only award the final **R1** if the first three marks have been awarded.

[5 marks]

Total [7 marks]

9. $y = vx,\ \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

$x\dfrac{dv}{dx} + v = \dfrac{x - vx}{x + vx}\left(= \dfrac{1-v}{1+v}\right)$

$x\dfrac{dv}{dx} + v = \dfrac{1-v}{1+v}$ ***(A1)***

$x\dfrac{dv}{dx} = \dfrac{1-v}{1+v} - v\left(= \dfrac{1-2v-v^2}{1+v}\right)$

attempt to separate variables and form two integrals ***(M1)***

$\int \dfrac{1+v}{1-2v-v^2}dv = \int \dfrac{1}{x}dx$

$\int \dfrac{1}{x}dx = \ln x + c$ ***A1***

use of substitution or inspection to integrate $\dfrac{1+v}{1-2v-v^2}$ or equivalent ***(M1)***

$u = 1-2v-v^2 \Rightarrow \dfrac{du}{dv} = -2-2v = -2(1+v)$

$\int \dfrac{1+v}{1-2v-v^2}dv = -\dfrac{1}{2}\ln\left|1-2v-v^2\right|$ OR $-\dfrac{1}{2}\ln\left|v^2+2v-1\right| = \ln|x| + c$ ***A1***

$-\dfrac{1}{2}\ln\left|1-2\dfrac{y}{x}-\dfrac{y^2}{x^2}\right| - \ln|x| = c$ OR $-\dfrac{1}{2}\ln\left|\dfrac{y^2}{x^2}+2\dfrac{y}{x}-1\right| - \ln|x| = c$

EITHER

attempt to substitute $x = 2$ and either $y = 0$ or $v = 0$ to find a constant c ***(M1)***

$c = -\ln 2$

$-\dfrac{1}{2}\ln\left|1-2\dfrac{y}{x}-\dfrac{y^2}{x^2}\right| - \ln|x| = -\ln 2$ OR $-\dfrac{1}{2}\ln\left|\dfrac{y^2}{x^2}+2\dfrac{y}{x}-1\right| - \ln|x| = -\ln 2$ ***A1***

OR

$x^2\left|1-2\dfrac{y}{x}-\dfrac{y^2}{x^2}\right| = A$ OR $x^2\left|\dfrac{y^2}{x^2}+2\dfrac{y}{x}-1\right| = A$ ***A1***

attempt to substitute $x = 2$ either $y = 0$ or $v = 0$ to find a constant A ***(M1)***

THEN

$x^2\left|1-2\dfrac{y}{x}-\dfrac{y^2}{x^2}\right| = 4$ OR $x^2\left(1-2\dfrac{y}{x}-\dfrac{y^2}{x^2}\right) = 4$ ***A1***

checking boundary values confirms $x^2 - 2xy - y^2 = 4$ ***AG***

continued…

Question 9 continued.

Note: Condone absence of absolute value signs even if removed incorrectly until the final ***A1*** mark where they must be seen or have been removed to form a correct equation.

Total [8 marks]

Section B

10. (a) **METHOD 1**

$a = 5$ ***(A1)***

attempt to use roots and symmetry to find h ***(M1)***

$h = \frac{(-1)+(-3)}{2}$ OR half the distance between the roots $\frac{(-1)-(-3)}{2} = 1$ (may be seen on a diagram)

$h = -2$ (accept $x = -2$) ***(A1)***

$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

($a = 5,\ h = -2,\ k = -5$)

METHOD 2

$a = 5$ ***(A1)***

attempt to expand

$(x+1)(x+3) = x^2 + 4x + 3$ OR $5(x+1)(x+3) = 5x^2 + 20x + 15$

EITHER

uses their expansion to attempt to complete the square to the form ***(M1)***

$p(x+q)^2 + r$,where q is half the coefficient of their x term

$= (x+2)^2 - 2^2 + 3\left(= (x+2)^2 - 1\right)$ OR $5\left[(x+2)^2 - 2^2 + 3\right](= 5(x+2)^2 - 5)$ ***(A1)***

OR

uses their expansion to attempt to differentiate and sets equal to zero ***(M1)***

$\frac{dy}{dx} = 2x + 4 = 0$ OR $\frac{dy}{dx} = 10x + 20 = 0$

$h = -2$ (accept $x = -2$) ***(A1)***

OR

uses their expansion to attempt to find axis of symmetry using $h = \frac{-b}{2a}$ ***(M1)***

$h = \frac{-4}{2}$ OR $h = \frac{-20}{10}$

$h = -2$ (accept $x = -2$) ***(A1)***

THEN

$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

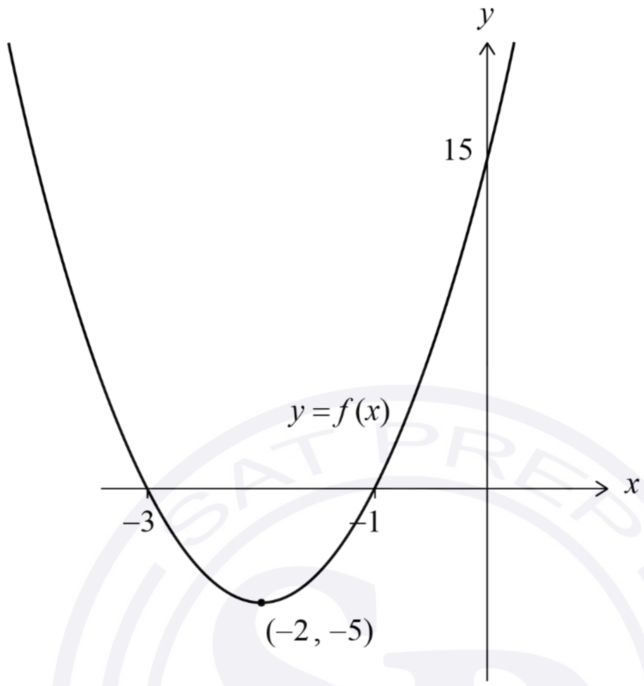
($a = 5,\ h = -2,\ k = -5$)

[4 marks]

continued…

Question 10 continued.

(b)



M1A1A1A1

award ***M1*** for a roughly symmetric curve which is concave up

award ***A1*** for x intercepts at -3 and -1

award ***A1*** for y intercept at 15

award ***A1*** for vertex at $(-2,-5)$

[4 marks]

(c) $5(x+2)^2 - 5 \le 40$ OR $5(x+1)(x+3) \le 40$ OR $(x+1)(x+3) \le 8$ leading to

$(x+2)^2 \le 9$ OR $5x^2 + 20x - 25 \le 0$ OR $x^2 + 4x - 5 \le 0$ ***(A1)***

valid attempt to find the critical values for their quadratic inequality ***(M1)***

$x + 2 = \pm 3$ OR $(x+5)(x-1) = 0$

$x = -5,\ x = 1$ ***(A1)***

$-5 \le x \le 1$ ***A1***

Note: Accept $(x \in)[-5,1]$ or equivalent.

[4 marks]

continued…

Question 10 continued.

(d) (i) $(f \circ g)(x) = 5(\ln x + 1)(\ln x + 3)$ OR $5(\ln x + 2)^2 - 5$ OR
$5(\ln x)^2 + 20\ln x + 15$ ***A1***

(ii) attempt to replace x with $\ln x$ using their solution to part (c) ***(M1)***

$-5 \le \ln x \le 1$

$e^{-5} \le x \le e$ ***A1***

Note: Accept $(x \in)[e^{-5}, e]$ or equivalent.

[3 marks]

(e) $(g \circ f)(x) = \ln(f(x))$

recognition that the domain requires $f(x) > 0$ ***(M1)***

$(x+1)(x+3) > 0$

$x < -3, x > -1$ ***A1A1***

Note: award **A1** for critical values and **A1** for correct inequalities. accept $(x \in)(-\infty, -3) \cup (-1, \infty)$ or equivalent.

[3 marks]

Total [18 marks]

11. (a) normal of Π_1 and Π_2 are $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ ***(A1)***

attempt to use the scalar product for the angle between two vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|\cos\theta$$

$-3=\sqrt{6}\sqrt{6}\cos\theta\ \ (\text{so } -3=6\cos\theta)$ ***A1A1***

Note: Award ***A1*** for correct scalar product and ***A1*** for correct magnitudes of normals and $\cos\theta$.

$\cos\theta=-\frac{1}{2}$ ***A1***

$\theta=120^\circ$ OR $\cos(180^\circ-\theta)=\frac{1}{2}$ ***A1***

acute angle is 60° ***AG***

[6 marks]

(b) attempt to find vector product of their normal vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}-4+1\\1+2\\-1-2\end{pmatrix}$$

$=\begin{pmatrix}-3\\3\\-3\end{pmatrix}\left(=3\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right)$ ***(A1)***

equation of Π_3 is $-3x+3y-3z=d$ OR $-x+y-z=d$ or equivalent

attempt to substitute $x=5,\ y=-5,\ z=5$ into their equation ***(M1)***

equation of Π_3 is $-3x+3y-3z=-45\ \ (\text{so } -x+y-z=-15)$ ***A1***

[4 marks]

continued…

Question 11 continued.

(c) (i) attempt to use the identity for $\sin\left(30^\circ + 45^\circ\right)$ ***(M1)***

$\sin\left(30^\circ + 45^\circ\right) = \sin\left(30^\circ\right)\cos\left(45^\circ\right) + \cos\left(30^\circ\right)\sin\left(45^\circ\right)$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ ***A1A1***

$= \frac{\sqrt{2} + \sqrt{6}}{4}$ ***AG***

> **Note:** ***A1*** *for each term.* Award ***A1 A0*** for correct answers where the denominator has not been rationalized such as $= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$.

(ii) let $x =$ QR

attempt to use the sine rule with the angles $75^\circ, 45^\circ$ ***(M1)***

$\frac{5}{\sin 75^\circ} = \frac{x}{\sin 45^\circ}$

$x = \frac{20}{\sqrt{2}\left(\sqrt{2} + \sqrt{6}\right)}\left(= \frac{20}{2\left(1 + \sqrt{3}\right)}\right)$ or equivalent ***A1***

attempt to rationalise a denominator of the form $\left(\sqrt{a} + \sqrt{b}\right)$ by multiplying numerator and denominator by $\pm\left(\sqrt{a} - \sqrt{b}\right)$ or equivalent ***(M1)***

$x = \frac{20\left(\sqrt{2} - \sqrt{6}\right)}{\sqrt{2}\left(\sqrt{2} + \sqrt{6}\right)\left(\sqrt{2} - \sqrt{6}\right)}\left(= \frac{20\left(1 - \sqrt{3}\right)}{2\left(1 + \sqrt{3}\right)\left(1 - \sqrt{3}\right)}\right)$

$x = 5\left(\sqrt{3} - 1\right)$ (cm) ($p = 5,\ q = 3$) ***A1***

[7 marks]

Total [17 marks]

12. (a) attempt to use integration by parts on $f_n(x) = \cos^{n-1} x \cos x$ ***M1***

$u = \cos^{n-1} x, \ \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = -(n-1)\cos^{n-2} x \sin x, \ v = \sin x$

$\int f_n(x)dx = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x dx$ ***A1A1***

Note: ***A1*** for each term with correct signs

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \left(1 - \cos^2 x\right) dx$ ***A1***

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$ ***AG***

[4 marks]

(b) attempt to rearrange the equation given in part (a) to collect terms in $\int f_n(x)dx$ or $\int \cos^n x dx$ ***M1***

$\int f_n(x)dx + (n-1)\int f_n(x)dx = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)dx$ OR
$\int \cos^n x dx + (n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$

$n\int f_n(x)dx = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)dx$ OR
$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$ ***A1***

$\int f_n(x)dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x)dx$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use equation from part (a) to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx$$

EITHER

attempt to use equation from part (a) again to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx\right)$$ ***(A1)***

OR

attempt to use double angle formula to rewrite $\cos^2 x$ in terms of $\cos 2x$ ***(M1)***

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int\left(\frac{\cos 2x + 1}{2}\right)dx$$ ***(A1)***

THEN

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + c$$ ***A1***

[4 marks]

(d) attempt to use the formula for volume of revolution using π and $\left(\cos^2 x\right)^2$ ***(M1)***

$$\text{volume} = \int \pi\left(\cos^2 x\right)^2 dx$$

$$\text{volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 dx \quad \text{OR} \quad 2\int_0^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 dx$$ ***(A1)***

Note: Condone omission dx for the ***A1***.

attempt to substitute correct limits into their (c) and subtract ***(M1)***

$$\pi\left(\frac{1}{4}\cos^3\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \pi\left(\frac{1}{4}\cos^3\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{3\pi^2}{8}$$ ***A1***

Note: ***FT*** marks may be awarded for a final answer of $r\pi^2$ based on non-zero values of p, q and r.

[4 marks]

continued…

Question 12 continued.

(e) (i) **METHOD 1**

attempt to raise Maclaurin expansion for $\cos x$ to the power of n ***(M1)***

$$f_n(x) = (\cos x)^n = \left(1 - \frac{x^2}{2} + \ldots\right)^n$$

$$= 1 - \frac{nx^2}{2} + \ldots$$ ***A2***

METHOD 2

attempt to differentiate $f_n(x)$ twice ***(M1)***

$$f_n{}'(x) = -n\cos^{n-1} x \sin x,\ f_n{}''(x) = -n\cos^n x + n(n-1)\cos^{n-2} x \sin^2 x$$ ***A1***

$$f_n(0) = 1,\ f_n{}'(0) = 0,\ f_n{}''(0) = -n$$

$$f_n(x) = 1 - \frac{nx^2}{2} + \ldots$$ ***A1***

(ii) **METHOD 1**

attempt to use Maclaurin expansion for $f_n(x)$ ***(M1)***

$$\lim_{x \to 0} \frac{f_n(x) - 1}{x^2} = \lim_{x \to 0} \frac{-\frac{nx^2}{2} + \ldots}{x^2} \left(= \lim_{x \to 0}\left(-\frac{n}{2} + \text{ powers of } x\right)\right)$$

$$= -\frac{n}{2}$$ ***A1***

METHOD 2

attempt to use l'Hôpital's rule twice on $\frac{\cos^n(x) - 1}{x^2}$ ***(M1)***

$$\lim_{x \to 0} \frac{\cos^n(x) - 1}{x^2} = \lim_{x \to 0} \frac{-n\cos^{n-1} x \sin x}{2x} = \lim_{x \to 0} \frac{n(n-1)\cos^{n-2} x \sin^2 x - n\cos^n x}{2}$$

$$= -\frac{n}{2}$$ ***A1***

Note: Do not award ***FT*** marks for an expression that does not involve n .

[5 marks]

Total [19 marks]

2225 – 7111M

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 1

31 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1****,* ***A2****.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full* ***FT*** *is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

SECTION A

1.

(a) -2 (accept $(0,-2)$) ***A1***

[1 mark]

(b) $y=\frac{3}{2}$ (must be an equation) ***A1***

Note: Do not accept $\neq$ sign.

[1 mark]

(c) attempt to reflect the graph of f in the x-axis OR one correct value seen ***(M1)***

both correct values $-\frac{3}{2}$ and 2 (seen anywhere) ***A1***

$-\frac{3}{2}<y\leq 2$ ***A1***

[3 marks]

Total [5 marks]

2. (a) attempt to set equal to a parameter or to add detail to cartesian form ***(M1)***

$\frac{x-1}{2}=\frac{y+2}{3}=z=\lambda \Rightarrow x=2\lambda+1, y=3\lambda-2, z=\lambda$ OR $\frac{x-1}{2}=\frac{y-(-2)}{3}=\frac{z-0}{1}$

$\boldsymbol{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r}=$ OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=$ is omitted.

[2 marks]

(b) attempt to equate at least one component ***(M1)***

$1+2\lambda=t$ OR $-2+3\lambda=4$ OR $\lambda=-8+2t$

$\lambda=2$ OR $t=5$ ***(A1)***

intersection point = $(5,4,2)$ ***A1***

Note: Condone coordinates given in vector form.

[3 marks]

Total [5 marks]

3.

(a) correct substitution in sine rule ***(A1)***

$\frac{\sin\theta}{5} = \frac{\sin 2\theta}{6\sqrt{2}}$ (or equivalent)

attempt to use double angle rule for $\sin 2\theta$ ***(M1)***

$\frac{\sin\theta}{5} = \frac{2\sin\theta\cos\theta}{6\sqrt{2}}$

$6\sqrt{2}\sin\theta = 10\sin\theta\cos\theta$ OR $\frac{1}{5} = \frac{2\cos\theta}{6\sqrt{2}}$ OR equivalent ***A1***

$\cos\theta = \frac{3\sqrt{2}}{5}$ ***AG***

[3 marks]

(b) valid attempt to find $\sin\theta$ ***(M1)***

$\sin^2\theta + \left(\frac{3\sqrt{2}}{5}\right)^2 = 1$ OR right triangle with adjacent side and hypotenuse labelled

$\sin\theta = \frac{\sqrt{7}}{5}$ ***A1***

[2 marks]

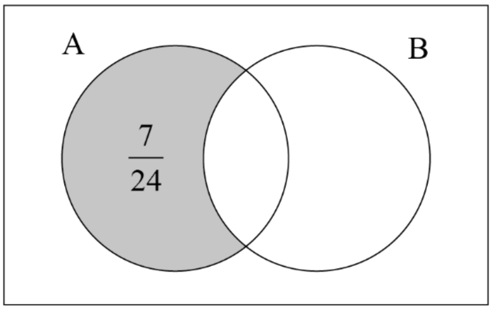
(c) $\frac{1}{2} \times 6\sqrt{2} \times \mathrm{DC} \times \frac{\sqrt{7}}{5} = 2\sqrt{14}$ ***(A1)***

$\mathrm{DC} = \frac{10}{3}$ ***A1***

[2 marks]

Total [7 marks]

4. (a) attempt to use a Venn diagram OR law of addition ***(M1)***



$$P(A \cup B) = P(A \cap B') + P(B)$$

$$\frac{5}{8} = \frac{7}{24} + P(B)$$ ***(A1)***

$$P(B) = \frac{8}{24}\left(= \frac{1}{3}\right)$$ ***A1***

[3 marks]

(b) **METHOD 1:** finding $P(A)$

attempt to find $P(A)$ ***M1***

$P(A \cap B') = P(A) \times P(B')$ OR $P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$P(A) \times \frac{2}{3} = \frac{7}{24}$ OR $\frac{5}{8} = \frac{2}{3}P(A) + \frac{1}{3}$

$$P(A) = \frac{7}{16}$$ ***(A1)***

EITHER

$P(A' \mid B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(A') \times P(B)}{P(B)} = P(A')$ (seen anywhere) OR $1 - \frac{7}{16}$ ***(A1)***

OR

$$P(A' \mid B)\left(= \frac{P(A' \cap B)}{P(B)}\right) = \frac{\frac{9}{48}}{\frac{16}{48}}\left(= \frac{\frac{3}{16}}{\frac{1}{3}}\right)$$ ***(A1)***

THEN

$$P(A' \mid B) = \frac{9}{16}$$ ***A1***

continued…

Question 4 continued.

METHOD 2: attempt to find $\text{P}(A \cap B)$

attempt to use $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$ ***M1***

$x = \left(x + \frac{7}{24}\right) \times \frac{1}{3}$

$x = \frac{7}{48}$ ***A1***

$\text{P}(A' \mid B) = \text{P}(B) - \text{P}(A \cap B) + 1 - \text{P}(A \cup B)$ OR $= \frac{1}{3} - \frac{7}{48} + 1 - \frac{5}{8}$ ***(A1)***

$\text{P}(A' \mid B) = \frac{27}{48} \left(= \frac{9}{16}\right)$ ***A1***

[4 marks]

Total [7 marks]

5.

(a) recognising $\Delta > 0$ (seen anywhere) ***(M1)***

$\Delta = k^2 - 4(15-k)$ $\left(= k^2 + 4k - 60\right)$ ***A1***

valid attempt to solve quadratic (in)equality ***(M1)***

$(k-6)(k+10)$ OR $k = \dfrac{-4 \pm \sqrt{4^2 - 4(-60)}}{2}$

two correct values -10 and 6 (seen anywhere) ***A1***

$k < -10,\ k > 6$ ***A1***

[5 marks]

(b) recognising constant or product of roots is positive ***(M1)***

$15 - k > 0 \Rightarrow k < 15$

$k < -10,\ 6 < k < 15$ ***A1***

[2 marks]

Total [7 marks]

6.

(a) correct substitution into distance formula ***A1***

$l^2 = x^2 \ln x + 4 - x^2 + x^2$ OR $\sqrt{(x-0)^2 + \left(\sqrt{x^2 \ln x + 4 - x^2} - 0\right)^2}$ OR

$\sqrt{x^2 + x^2 \ln x + 4 - x^2}$

$l = \sqrt{x^2 \ln x + 4}$ ***AG***

[1 mark]

(b) recognising $\frac{\mathrm{d}l}{\mathrm{d}x} = 0$ (seen anywhere) ***(M1)***

EITHER

attempt to use chain rule with l ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$

attempt to use product rule with $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$ ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \left[x^2 \times \frac{1}{x} + \ln x \times 2x\right]$ ***A1***

OR

recognising to minimise $x^2 \ln x + 4$ ***(M1)***

attempt to use product rule ***(M1)***

$x^2 \times \frac{1}{x} + \ln x \times 2x$ ***A1***

THEN

$x + 2x \ln x = 0$ (or equivalent) ***A1***

$\Rightarrow \ln x = -\frac{1}{2}$

$x = \mathrm{e}^{-\frac{1}{2}} \quad \left(= \frac{1}{\sqrt{\mathrm{e}}}\right)$ ***A1***

Note: Award ***A0*** for including $x = 0$ in the final answer.

[6 marks]

Total [7 marks]

7. **METHOD 1** (product of quadratic factors)

attempt to write as product of two quadratic factors ***(M1)***

$x^4 + px^3 - 2x^2 + qx - 3 = \left(x^2 + 2x + 1\right)\left(ax^2 + bx + c\right)$

$= \left(x^2 + 2x + 1\right)\left(x^2 + bx - 3\right)$ OR $\left(x^2 + 2x + 1\right)\left(x^2 - 3\right)$ ***A1***

attempt to compare their coefficients ***(M1)***

$p = b + 2$ OR $q = b - 6$ OR $-2 = 2b + 1 - 3$

$b = 0$ ***(A1)***

$p = 2$, $q = -6$ ***A1***

METHOD 2 (double root)

let $f(x) = x^4 + px^3 - 2x^2 + qx - 3$

recognition that $f(-1) = 0$ ***(M1)***

$p + q = -4$ ***A1***

recognition that $f'(x) = 0$ at $x = -1$ since it has a double root ***(M1)***

$f'(x) = 4x^3 + 3px^2 - 4x + q$

$3p + q = 0$ ***A1***

$p = 2$, $q = -6$ ***A1***

METHOD 3 (division by $(x+1)$ twice)

attempt to use division with $(x+1)$ to find remainder ***(M1)***

$p + q = -4$ ***A1***

attempt to use division once again with $(x+1)$ to find remainder ***(M1)***

$q + 3p = 0$ ***A1***

$p = 2$, $q = -6$ ***A1***

METHOD 4 (division by $x^2 + 2x + 1$)

attempt to use division to find remainder ***(M1)***

$(q + 3p)x + 2p - 4(= 0)$ (or equivalent) ***(A1)***

equating the coefficients of their remainder to 0 ***(M1)***

$q + 3p = 0$ and $2p - 4 = 0$ OR $q - p + 2 = 2(1 - 2p)$ and $-3 = 1 - 2p$ (or equivalent) ***A1***

$p = 2$, $q = -6$ ***A1***

continued…

Question 7 continued.

METHOD 5 (sum and product of roots)

let the four roots be $-1, -1, \alpha$ and β

attempt to set sum of four roots equal to $-p$ OR product of four roots equal to -3 ***(M1)***

Note: Award ***M1*** for expansion of $(x-1)^2(x-\alpha)(x-\beta)$ leading to a quartic $x^4+(2-\alpha-\beta)x^3+(\alpha\beta-2\alpha-2\beta+1)x^2+(2\alpha\beta-\alpha-\beta)x+\alpha\beta\ (=0)$

$(-1)+(-1)+\alpha+\beta=-p\ (\Rightarrow \alpha+\beta=-p+2)$ and $(-1)(-1)\alpha\beta=-3\ (\Rightarrow \alpha\beta=-3)$ ***A1***

sum of product of pairs and sum of product of roots taken three at a time

$\alpha\beta-\alpha-\alpha-\beta-\beta+1=-2\ (\Rightarrow \alpha\beta-2\alpha-2\beta=-3)$ and

$-\alpha\beta-\alpha\beta+\beta+\alpha=-q\ (\Rightarrow -2\alpha\beta+\beta+\alpha=-q)$ ***A1***

Note: Award ***A1*** for any two correct equations.

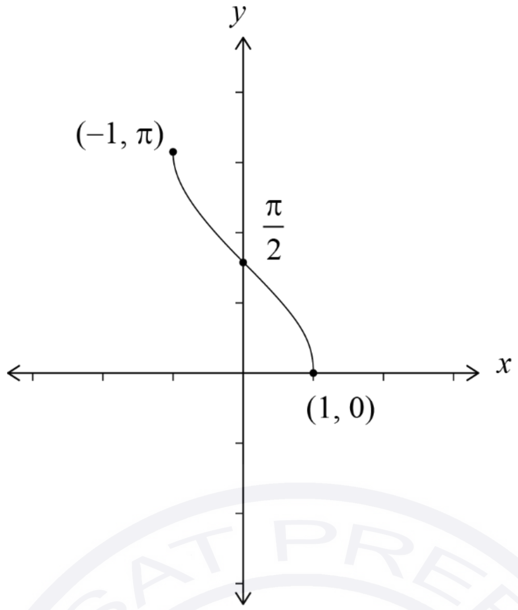
attempt to solve *their* four equations to obtain the value of p and q ***(M1)***

$\alpha\beta=-3$ and $\alpha\beta-2\alpha-2\beta=-3 \Rightarrow \alpha+\beta=0$

$p=2$, $q=-6$ ***A1***

[5 marks]

8. (a)



For a curve with correct shape within the domain, correct labelled y-intercept and end points. ***A2***

Note: Award ***A1*** for any two of the following provided a curve is seen: correct shape within the domain correctly labelled y - intercept correctly labelled end points.

[2 marks]

continued…

Question 8 continued.

(b) **METHOD 1**

recognizing to use compound angle rule for $\cos(A+B)$ ***(M1)***

$\cos(\arccos x)\cos\left(\arccos x\sqrt{3}\right)-\sin(\arccos x)\sin\left(\arccos x\sqrt{3}\right)$

valid attempt to find $\sin(\arccos x)$ or $\sin\left(\arccos x\sqrt{3}\right)$ in terms of x ***(M1)***

use of the identity $\sin\theta=\sqrt{1-\cos^2\theta}$ OR use of a right-angled triangle

($\sin A$ and $\sin B$ are both positive)

$x\times x\sqrt{3}-\sqrt{1-x^2}\times\sqrt{1-3x^2}\left(=\cos\frac{3\pi}{2}\right)$ ***A1***

$\cos\frac{3\pi}{2}=0$ (seen anywhere) ***(A1)***

attempt to write an equation in x without the square roots ***(M1)***

$3x^4=1-4x^2+3x^4$

$x=\pm\frac{1}{2}$

(as seen in part (a) graph, when $x=\frac{1}{2}$, $\arccos(x)+\arccos\left(x\sqrt{3}\right)<\pi\Rightarrow$ no solution)

$x=-\frac{1}{2}$ ***A1***

continued…

Question 8 continued.

METHOD 2

recognizing to use compound angle rule for $\sin(A+B)$ ***(M1)***

$\sin(\arccos x)\cos\left(\arccos x\sqrt{3}\right)+\cos(\arccos x)\sin\left(\arccos x\sqrt{3}\right)$

valid attempt to find $\sin(\arccos x)$ or $\sin\left(\arccos x\sqrt{3}\right)$ in terms of x ***(M1)***

use of the identity $\sin\theta=\sqrt{1-\cos^2\theta}$ OR use of a right-angled triangle

($\sin A$ and $\sin B$ are both positive)

$\sqrt{1-x^2}\times x\sqrt{3}+x\times\sqrt{1-3x^2}\left(=\sin\frac{3\pi}{2}\right)$ ***A1***

$\sin\frac{3\pi}{2}=-1$ (seen anywhere) ***(A1)***

attempt to write an equation in x without the square roots ***(M1)***

$4x^4-4x^2+1=4x^2-12x^4\ \left(\Rightarrow 16x^4-8x^2+1=0\Rightarrow\left(4x^2-1\right)^2=0\right)$

$x=\pm\frac{1}{2}$

(as seen in part (a) graph, when $x=\frac{1}{2}$, $\arccos(x)+\arccos\left(x\sqrt{3}\right)<\pi\Rightarrow$ no solution)

$x=-\frac{1}{2}$ ***A1***

[6 marks]

Total [8 marks]

9. METHOD 1

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

> **Note:** Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".
>
> Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \le 4$".

since $0 < x < 1$ ($\Rightarrow x > 0$ and $1 - x > 0$, therefore) $x(1-x) > 0$ (seen anywhere) ***R1***

> **Note:** Subsequent marks after this ***M1R1*** are independent of these two marks and can only be awarded provided a correct or weak inequality ($<$ or $\le$) is used at the assumption step.

attempt to form a quadratic inequality in x (using their assumption) ***(M1)***

$1 < 4x(1-x)$ OR $\frac{1-4x(1-x)}{x(1-x)} < 0$

$4x^2 - 4x + 1 < 0$ (or equivalent) ***A1***

EITHER

$(2x-1)^2 < 0$ (or equivalent) ***A1***

a contradiction, since $(2x-1)^2 \ge 0$ (for $0 < x < 1$) ***R1***

OR

a graph of $y = 4x^2 - 4x + 1$ with the vertex touching the x-axis at $x = \frac{1}{2}$ ***A1***

a contradiction, since $4x^2 - 4x + 1 \ge 0$ (for $0 < x < 1$) ***R1***

THEN

hence $\frac{1}{x(1-x)} \ge 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

> **Note:** Award a maximum of ***M0R1(M1)A1A1R0*** for incorrect use of weak inequality signs.

continued…

Question 9 continued.

METHOD 2

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

Note: Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".

Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \le 4$".

Subsequent marks after this ***M1*** are independent and can only be awarded provided a correct or weak inequality ($<$ or $\le$) is used at the assumption step.

$\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x} \Rightarrow 1 \equiv A(1-x) + Bx$

attempt to equate both coefficients OR substitute two values eg 0 and 1 ***(M1)***

$A = 1$ and $B = 1$ ***A1***

Note: Award ***A1*** for both correct values.

Let $f(x) = \frac{1}{x} + \frac{1}{1-x}$

equating their $f'(x) = 0$ and solve for x OR a concave upward graph for $0 < x < 1$ ***(M1)***

$f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = 0 \Rightarrow x^2 - 2x + 1 = x^2 \Rightarrow x = \frac{1}{2}$

minimum value is $f\left(\frac{1}{2}\right) = 4$ ***A1***

a contradiction, since $4 < 4$ ***R1***

hence $\frac{1}{x(1-x)} \ge 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

Note: Award a maximum of ***M0(M1)A1(M1)A1R0*** for incorrect use of weak inequality signs.

continued…

Question 9 continued.

METHOD 3

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

> **Note:** Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".
>
> Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \le 4$".
>
> Subsequent marks after this ***M1*** are independent and can only be awarded provided a correct or weak inequality ($<$ or $\le$) is used at the assumption step.

Let $f(x) = \frac{1}{x(1-x)}$

attempt to use product rule to find $f'(x)$ ***(M1)***

$f'(x) = -\frac{1}{x^2(1-x)} + \frac{1}{x(1-x)^2} \left(= \frac{2x-1}{x^2(1-x)^2} \right)$ ***A1***

equating their $f'(x) = 0$ and solve for x ***(M1)***

$x = \frac{1}{2}$

minimum value is $f\left(\frac{1}{2}\right) = 4$ ***A1***

a contradiction, since $4 < 4$ ***R1***

hence $\frac{1}{x(1-x)} \ge 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

> **Note:** Award a maximum of ***M0(M1)A1(M1)A1R0*** for incorrect use of weak inequality signs.

[6 marks]

Section B

10.

(a) $4^x = 8$ OR $f^{-1}(x) = \log_4 x$ OR $f^{-1}(8) = \log_4 8$ ***(A1)***

attempt to use indices with same base OR change of base of logs ***(M1)***

$2^{2x} = 2^3,\ 4^x = 4^{\frac{3}{2}}$ OR $f^{-1}(8) = \log_4 4^{\frac{3}{2}}$ OR $f^{-1}(8) = \dfrac{\log_2 8}{\log_2 4}$

$f^{-1}(8) = \dfrac{3}{2}$ ***A1***

[3 marks]

(b) (i) interchanging x and y (seen anywhere) ***(M1)***

$x = 1 + \log_2 y$ OR $y - 1 = \log_2 x$

$x - 1 = \log_2 y$ OR $2^{y-1} = x$

$g^{-1}(x) = 2^{x-1}$ (or equivalent) ***A1***

(ii) **METHOD 1**

a horizontal translation/shift by 1 unit to the left (do not accept 'move')

followed by a horizontal stretch/dilation with scale factor $\dfrac{1}{2}$ (accept horizontal compression by a factor of 2)

one correct transformation ***A1***

two correct transformations in the correct order ***A1***

METHOD 2

horizontal stretch/dilation with scale factor $\dfrac{1}{2}$ (accept horizontal compression by a factor of 2) ***A1***

vertical stretch/dilation with scale factor 2 ***A1***

Note: ***A1***s can be awarded independently here as order is not important.

[4 marks]

continued…

Question 10 continued.

(c) attempt to find composite function (in any order) ***M1***

$f(1+\log_2 x)\left(=4^{1+\log_2 x}\right)$

$4\times4^{\log_2 x}$ OR $4\times2^{2\log_2 x}$ OR $4^{\log_2 2x}$ OR $2^{(2+2\log_2 x)}$ OR $4^{(\log_4 4+2\log_4 x)}$ ***(A1)***

$=4\times2^{\log_2 x^2}$ OR $2^{\log_2 (2x)^2}$ OR $4^{\log_4 (4x^2)}$ ***A1***

$=4x^2$ ***AG***

[3 marks]

(d) (i) $2x-1+\dfrac{1}{2x+1}=\dfrac{(2x-1)(2x+1)+1}{2x+1}$ ***(A1)***

$=\dfrac{4x^2-1+1}{2x+1}$ ***A1***

$2x-1+\dfrac{1}{2x+1}=\dfrac{4x^2}{2x+1}$ ***AG***

Note: Accept working from RHS to LHS.

(ii) **METHOD 1**

enclosed area is $\int_1^3\left(2x-1+\dfrac{1}{2x+1}\right)\mathrm{d}x$

attempt to integrate (at least one correct term or $\ln(2x+1)$ seen) ***(M1)***

$=x^2-x+\dfrac{1}{2}\ln|2x+1|(+c)$ ***A1A1***

Note: Award ***A1*** for x^2-x and ***A1*** for $\frac{1}{2}\ln\lvert 2x+1\rvert$. Accept $\frac{1}{2}\ln(2x+1)$.

substitute correct limits into their integrated expression and subtract ***(M1)***

$=\left(9-3+\dfrac{1}{2}\ln 7\right)-\left(1-1+\dfrac{1}{2}\ln 3\right)$

$A=6+\dfrac{1}{2}\ln\dfrac{7}{3}$ ***A1***

continued…

Question 10 continued.

METHOD 2

attempt to use integration by substitution ***(M1)***

let $u = 2x+1$ OR $u = 2x-1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$= \frac{1}{2}\int \left(u - 2 + \frac{1}{u}\right)\mathrm{d}u$ OR $\frac{1}{2}\int \frac{(u-1)^2}{u}\mathrm{d}u$ OR $\frac{1}{2}\int \left(u + \frac{1}{u+2}\right)\mathrm{d}u$ ***A1***

correct integration

$= \frac{1}{4}u^2 - u + \frac{1}{2}\ln|u|(+c)$ OR $\frac{1}{4}u^2 + \frac{1}{2}\ln|u+2|(+c)$ ***A1***

substitution of their limits into their integrated expression and subtract ***(M1)***

$= \left(\frac{49}{4} - 7 + \frac{1}{2}\ln 7\right) - \left(\frac{9}{4} - 3 + \frac{1}{2}\ln 3\right)$ OR $\left(\frac{25}{4} + \frac{1}{2}\ln 7\right) - \left(\frac{1}{4} + \frac{1}{2}\ln 3\right)$

$A = 6 + \frac{1}{2}\ln\frac{7}{3}$ (or equivalent) ***A1***

[7 marks]

Total [17 marks]

11. (a) attempt to use the binomial expansion with $n = \frac{1}{2}$ ***(M1)***

$= 1 + \left(\frac{1}{2}\right)(5x) + \ldots$

correct substitution into at least three terms ***(A1)***

$$= 1 + \left(\frac{1}{2}\right)(5x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(5x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(5x)^3 + \ldots$$

$= 1 + \frac{5}{2}x - \frac{25}{8}x^2 + \frac{125}{16}x^3 \ (+\ldots)$ ***A1A1***

Note: Award ***A1*** for $1 + \frac{5}{2}x - \frac{25}{8}x^2$ and ***A1*** for $\frac{125}{16}x^3$.

[4 marks]

(b) at least 3 correct terms in expansion of $(1+qx)^{-1}$ ***A1***

$= 1 + (-1)(qx) + \frac{(-1)(-2)}{2!}(qx)^2 + \ldots \left(= 1 - qx + q^2x^2 + \ldots\right)$

attempt to find product ***(M1)***

$(1+px)\left(1 - qx + q^2x^2 + \ldots\right) = 1 - qx + q^2x^2 + \ldots + px - pqx^2 + \ldots$

$= 1 + px - qx + q^2x^2 - pqx^2 + \ldots\left(= 1 + (p-q)x + \left(q^2 - pq\right)x^2 + \ldots\right)$ ***A1***

Note: Award ***A0*** if the final answer is not in ascending order.

[3 marks]

continued…

Question 11 continued.

(c) equating their coefficients of either x or x^2 ***(M1)***

one correct equation ***A1***

$p-q=\frac{5}{2}$ OR $q^2-pq=-\frac{25}{8}$

valid attempt to solve for q or p ***(M1)***

$-\frac{5}{2}q=-\frac{25}{8}$ OR $q^2-q\left(q+\frac{5}{2}\right)=-\frac{25}{8}$ OR $\frac{15}{4}-q=\frac{5}{2}$ ***A1***

Note: Award full marks for $\frac{15}{4}-q=\frac{5}{2}$ provided a full solution for $p=\frac{15}{4}$ is seen first. Using $p=\frac{15}{4}$ from part (d)'s stem is not sufficient.

$q=\frac{5}{4}$ ***AG***

[4 marks]

(d) (i) attempt to find x ***(M1)***

$1+5x=1.2$

$x=\frac{1}{25}(=0.04)$ (or equivalent) ***A1***

substituting their value of x with p and q ***(M1)***

$$\frac{1+\frac{15}{4}\left(\frac{1}{25}\right)}{1+\frac{5}{4}\left(\frac{1}{25}\right)}$$

$\frac{115}{105}$ $\left(=\frac{23}{21}\right)$ ***A1***

Note: Award ***(M1)A1(M1)A1*** for an answer of $\frac{219}{200}$ or $\frac{2191}{2000}$ using the expansion from part (b) or (a).

continued…

Question 11 continued.

(ii) $\frac{\sqrt{5}}{2} = \sqrt{\frac{5}{4}} = \sqrt{1.25}$

$x = \frac{1}{20}$ ***A1***

recognising $\frac{1}{20} > \frac{1}{25}$ ***R1***

therefore, approximation for $\frac{\sqrt{5}}{2}$ is not as accurate as the approximation

for $\sqrt{1.2}$ ***AG***

[6 marks]

Total [17 marks]

12. (a) $z^2 = 2\left(\cos\left(-\frac{2\pi}{3}\right) + \text{i}\sin\left(-\frac{2\pi}{3}\right)\right)$ OR $z^2 = 2\left(\cos\left(\frac{4\pi}{3}\right) + \text{i}\sin\left(\frac{4\pi}{3}\right)\right)$ ***A2***

Note: Award ***A1*** for correct modulus or argument seen.

$z^2 = 2\left(\cos\left(-\frac{2\pi}{3} + 2k\pi\right) + \text{i}\sin\left(-\frac{2\pi}{3} + 2k\pi\right)\right)$ OR $z^2 = 2\left(\cos\left(\frac{4\pi}{3} + 2k\pi\right) + \text{i}\sin\left(\frac{4\pi}{3} + 2k\pi\right)\right)$

attempt to use De Moivre's theorem ***M1***

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{3} + k\pi\right) + \text{i}\sin\left(-\frac{\pi}{3} + k\pi\right)\right)$ OR $z = \sqrt{2}\left(\cos\left(\frac{2\pi}{3} + k\pi\right) + \text{i}\sin\left(\frac{2\pi}{3} + k\pi\right)\right)$

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + \text{i}\sin\left(-\frac{\pi}{3}\right)\right)$; $z = \sqrt{2}\left(\cos\left(\frac{2\pi}{3}\right) + \text{i}\sin\left(\frac{2\pi}{3}\right)\right)$ ***A1***

Note: accept $z = \sqrt{2}\left(\cos\left(\frac{5\pi}{3}\right) + \text{i}\sin\left(\frac{5\pi}{3}\right)\right)$ in place of the 1st term above. accept $r\,\text{cis}\,\theta$ form throughout.

[4 marks]

(b) (i) $z_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\text{i}$ and $z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\text{i}$ ***A1A1***

Note: may be seen in part (a). Award ***A1A0*** for z_1 and z_2 interchanged. Accept factorised form, e.g. $z_1 = \sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}\right)$.

(ii) $z_3 = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\text{i}$ and $z_4 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\text{i}$ ***A1A1***

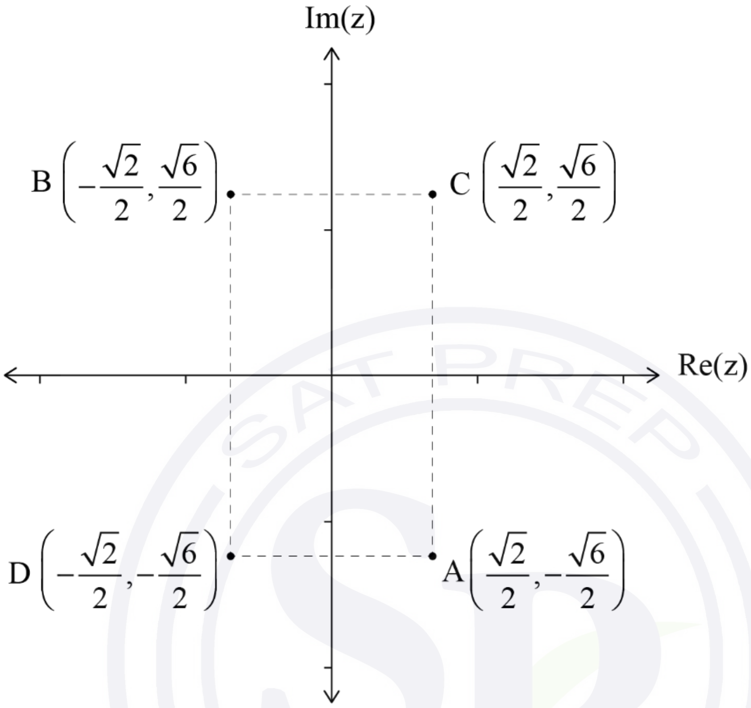
Note: Award ***A1A0*** for z_3 and z_4 interchanged. Accept factorised form, e.g. $z_3 = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right)$. Award ***A1FTA1FT*** for $z_3 = their\left(z_1\right)^*$ with $\text{Re}\left(z_3\right) > 0$ and $z_4 = their\left(z_2\right)^*$.

[4 marks]

continued…

Question 12 continued.

(c) (i) Argand diagram with all four points plotted in approximately correct positions forming a rectangle. ***A2***



Note:	Award ***A1*** for any two points plotted correctly on an argand diagram.
	Award ***A1A0*** if the four points form a square.
	Condone absence of labels, coordinates if they are approximately in correct positions and same distance from the origin. Dotted lines not required.

(ii) **METHOD 1**

attempt to find the length OR the width of rectangle ***(M1)***

$2\times\frac{\sqrt{2}}{2}$ OR $2\times\frac{\sqrt{6}}{2}$

length $=\sqrt{6}$ and width $=\sqrt{2}$

area of rectangle $=\sqrt{12}\ \left(=2\sqrt{3}\right)$ ***A1***

continued…

Question 12 continued.

METHOD 2

attempt to find the area of triangle ***(M1)***

$\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\sin\frac{2\pi}{3}$ OR $\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\sin\frac{\pi}{3}$

$4\times\left(\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\frac{\sqrt{3}}{2}\right)$

area of rectangle $=2\sqrt{3}$ ***A1***

[4 marks]

(d) recognition that $z=\frac{1}{w}$ ***(M1)***

$\left(\frac{1}{w}\right)^4+2\left(\frac{1}{w}\right)^2+4=0$ OR $\frac{1+2w^2+4w^4}{w^4}=0$ ***(A1)***

$4w^4+2w^2+1=0$

$p=4,q=2,r=1$ ***A1***

[3 marks]

(e) (i) **EITHER**

attempt to take conjugate ***M1***

$$\frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\mathrm{i}\right)}\times\frac{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}\right)}{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}\right)} \text{ OR } \frac{2}{\left(\sqrt{2}-\sqrt{6}\,\mathrm{i}\right)}\times\frac{\left(\sqrt{2}+\sqrt{6}\,\mathrm{i}\right)}{\left(\sqrt{2}+\sqrt{6}\,\mathrm{i}\right)}$$

OR

attempt to use De Moivre's theorem ***M1***

$$\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$$

THEN

$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\mathrm{i}$ (or equivalent) ***A1***

continued…

Question 12 continued

(ii) recognition of dilation/stretch by a scale factor of $\frac{1}{2}$ for both dimensions ***(M1)***

area of rectangle $=\frac{\sqrt{12}}{4}\left(=\frac{\sqrt{3}}{2}\right)$ ***A1***

Note: Award follow through from their area found in part (c)(ii).

[4 marks]

Total [19 marks]

2225 – 7116M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 1

23 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

  Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10 Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) correct application of $\log_a xy = \log_a x + \log_a y$ or $\log_a x^m = m\log_a x$ ***(M1)***

correct expression in terms of $\log_{10} 2$ AND $\log_{10} 3$ (arguments must be 2 and 3) ***(A1)***

$3\log_{10} 2 + \log_{10} 3$ OR $\log_{10} 2 + \log_{10} 2 + \log_{10} 2 + \log_{10} 3$

$3p + q$ ***A1***

[3 marks]

(b) $\left(\log_3 8 =\right) \dfrac{\log_{10} 8}{\log_{10} 3} \left(= \dfrac{3\log_{10} 2}{\log_{10} 3}\right)$ ***(A1)***

$= \dfrac{3p}{q}$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $f(-3) = -1$ ***A1***

[1 mark]

(b) $-3 \le x \le 5$ ***A1***

Note: Award ***A1*** for answers using interval notation $[-3,\ 5]$.

[1 mark]

(c) $\left(f^{-1}(2x-7) = -3 \Rightarrow\right) 2x - 7 = f(-3)$ **OR** $f^{-1}(-1) = -3$ ***(M1)***

$2x - 7 = -1$ ***(A1)***

$x = 3$ ***A1***

[3 marks]

Total [5 marks]

3. recognizing to use $\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$2\left(2\cos^2\theta - 1\right) - 5\cos\theta + 2 \ (= 0)$ ***A1***

$4\cos^2\theta - 5\cos\theta \ (= 0)$

choosing an appropriate method to solve their quadratic equation ***(M1)***

$\cos\theta(4\cos\theta - 5)$ **OR** $\dfrac{5 \pm \sqrt{(-5)^2 - 4\times 4\times 0}}{2\times 4}$ ***(A1)***

$\cos\theta = 0$

$\theta = \dfrac{3\pi}{2}$ ***A1***

Note: Do not award final ***A1*** if any extra solutions given.

[5 marks]

4. (a) equating $y = mx - 3$ and $y = x^2 - x - 1$ ***M1***

$x^2 - x - 1 - (mx - 3) = 0$ **OR** $x^2 - mx - x + 2 = 0$ ***A1***

$x^2 - (m+1)x + 2 = 0$ ***AG***

[2 marks]

(b) **METHOD 1 (discriminant)**

correct substitution into discriminant (do not award if seen only in quadratic formula) ***(A1)***

$\left(-(m+1)\right)^2 - 4(1)(2)$ **OR** $(m+1)^2 - 4(1)(2)$ ***A1***

discriminant equals 0 (seen anywhere) ***A1***

$(m+1)^2 = 8$ **OR** $m^2 + 2m - 7 = 0$

$(m =)\ -1 - 2\sqrt{2},\ -1 + 2\sqrt{2}$ ***A1A1***

METHOD 2 (derivative)

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - 1$ ***A1***

substituting $m = 2x - 1$ or $x = \dfrac{m+1}{2}$ into ***AG*** from part (a) ***(A1)***

$x^2 - (2x - 1 + 1)x + 2 = 0$ **OR** $x^2 - 2x^2 + 2 = 0$ **OR** $\left(\dfrac{m+1}{2}\right)^2 - (m+1)\dfrac{m+1}{2} + 2 = 0$

$x = \pm\sqrt{2}$ **OR** $-m^2 - 2m + 7 = 0$ **OR** $m^2 + 2m - 7 = 0$ ***A1***

$(m =)\ 2\sqrt{2} - 1,\ -2\sqrt{2} - 1$ ***A1A1***

[5 marks]

Total [7 marks]

5. (a) recognizing constant change from Y to X and/or comparing the two distributions

$b-7=22-19$ **OR** $b=7+(22-19)$ **OR** $\frac{b-7}{a}=\frac{22-19}{a}$ ***(M1)***

$b=10$ ***A1***

[2 marks]

(b) $\left(\mathrm{P}(7-a<X<7+a)=\right)\ 0.68$ ***A1***

[1 mark]

(c) **EITHER**

recognizing that 22 is one standard deviation above the mean ***(M1)***

OR

recognizing symmetry of the normal curve and total area $=1$ ***(M1)***

THEN

0.16 or 0.34 (or equivalent) seen in correct sketch or probability statement ***(A1)***

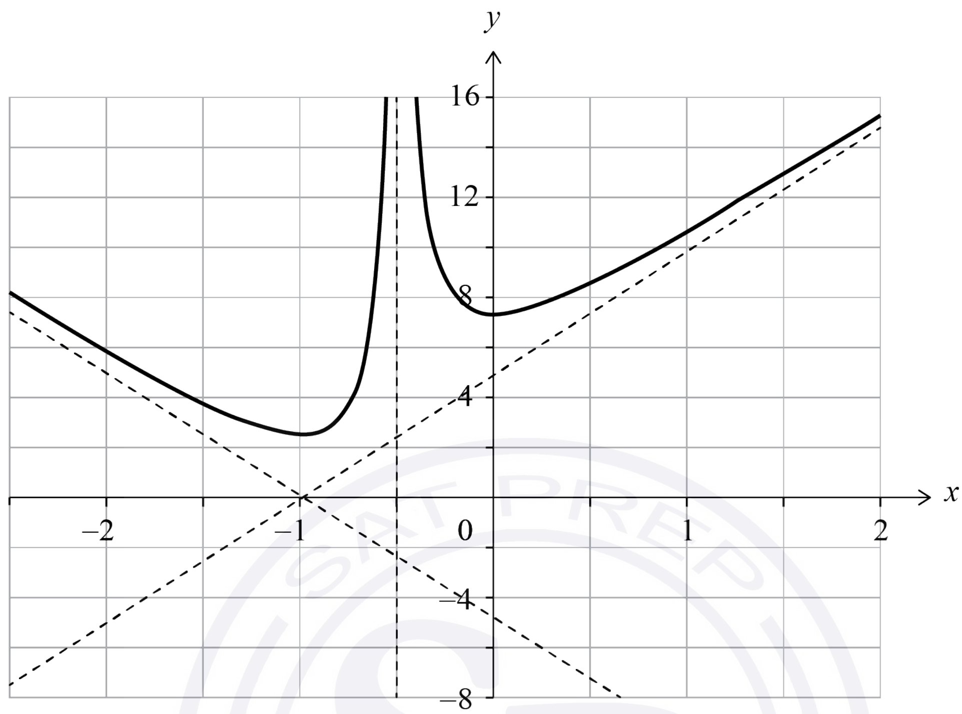
$\left(\mathrm{P}(Y<22)=\right)1-\frac{0.32}{2}$ **OR** $0.5+0.34$ **OR** $0.68+0.16$

0.84 ***A1***

[3 marks]

Total [6 marks]

6. (a)



right branch with minimum in approximately correct position **AND** showing asymptotic behaviour to $y = 5x + 5$ ***A1***

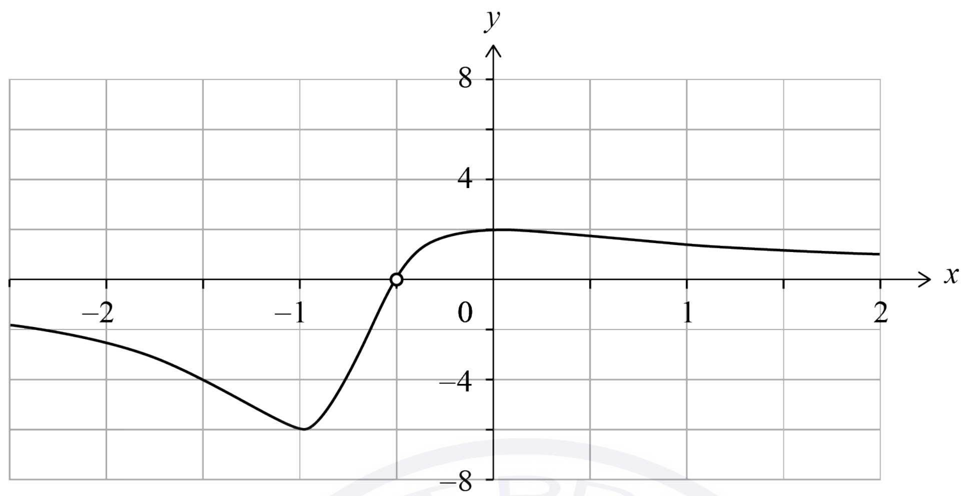
asymptotic behaviour on both branches at $x = -\frac{1}{2}$ ***A1***

left branch reflected in x-axis with minimum in approximately correct position ***M1***

oblique asymptote ($y = -5x - 5$) drawn in approximately correct position (equation is not required) **AND** left branch with correct asymptotic behaviour ***A1***

[4 marks]

(b)



axes intercepts in approximately correct positions ***A1***

local extrema in approximately correct positions ***A1***

asymptotic behavior to $y = 0$ on both sides **OR** asymptote at $y = 0$ indicated on sketch (by line or label etc. equation not required) ***A1***

[3 marks]

Total [7 marks]

7. $x^2 = r^4 - y^4$ (seen anywhere) ***(A1)***

substituting their expression for x^2 into $\int_a^b \pi x^2 \mathrm{d}y$ ***(M1)***

$\pi\int_{-r}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ **OR** $2\pi\int_{0}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ ***(A1)***

$= \pi\left[r^4 y - \frac{y^5}{5}\right]_{-r}^{r}$ **OR** $2\pi\left[r^4 y - \frac{y^5}{5}\right]_{0}^{r}$ ***A1***

Note: Award ***A1*** for $r^4 y - \frac{y^5}{5}$ seen.

$= \pi\left(\left(r^5 - \frac{r^5}{5}\right) - \left(-r^5 + \frac{r^5}{5}\right)\right)$ **OR** $2\pi\left(r^5 - \frac{r^5}{5}\right)$ ***(A1)***

$= \frac{8}{5}\pi \mathrm{r}^5$ ***A1***

Total [6 marks]

8. (a) attempt to find modulus **OR** argument of z_1 ***(M1)***

$r = \sqrt{12}$ $\left(= 2\sqrt{3}\right)$ or $\theta = -\frac{\pi}{3}$ ***A1***

$z_1 = \sqrt{12}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ $\left(= 2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)$ ***A1***

[3 marks]

(b) $\left(\frac{z_2}{z_1} = \frac{2\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}}{2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}} =\right) \mathrm{e}^{\mathrm{i}\frac{7\pi}{6}} \left(= \mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}\right)$ ***(A1)***

recognizing three equivalent arguments ***(M1)***

eg $\frac{z_2}{z_1} = \mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6} + 2\pi k\right)}$ **OR** $\frac{z_2}{z_1} = \mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}, \mathrm{e}^{-\frac{17\pi}{6}\mathrm{i}}, \mathrm{e}^{\frac{7\pi}{6}\mathrm{i}}$

attempt to find a root using de Moivre's theorem ***(M1)***

$\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{18} + \frac{2\pi k}{3}\right)} \left(= \mathrm{e}^{\mathrm{i}\left(\frac{(12k-5)\pi}{18}\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{(12k+7)\pi}{18}\right)}\right)$ ***(A1)***

$\mathrm{e}^{-\mathrm{i}\frac{17\pi}{18}}, \mathrm{e}^{-\mathrm{i}\frac{5\pi}{18}}, \mathrm{e}^{\mathrm{i}\frac{7\pi}{18}}$ ***A1***

[5 marks]

Total [8 marks]

9. METHOD 1

$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***R1***

Note: Subsequent marks are independent of this ***R*** mark.

$=\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}\ \left(=\lim_{x\to 0}\left(1+\frac{x\cos x}{\sin x}\right)\right)$ ***A1***

EITHER

Applying l'Hôpital's rule again (since $\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\frac{0}{0}$) ***(M1)***

Note: Award ***(M0)*** if their limit is not the indeterminate form $\frac{0}{0}$.

$\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}$ ***A1***

OR

$=\lim_{x\to 0}\left(1+\frac{\cos x}{\left(\frac{\sin x}{x}\right)}\right)$ ***A1***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ ***R1***

THEN

substituting $x=0$ ***(M1)***

$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$ ***A1***

continued…

Question 9 continued

METHOD 2

multiplying numerator and denominator by $1+\cos x$ ***M1***

$$\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$$

$$\lim_{x\to 0}\left(\frac{x\sin x(1+\cos x)}{1-\cos^2 x}\right)$$

$$\lim_{x\to 0}\left(\frac{x(1+\cos x)}{\sin x}\right)$$ ***A1***

$$=\lim_{x\to 0}\left(\frac{1+\cos x}{\left(\frac{\sin x}{x}\right)}\right)$$ ***(A1)***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ OR $\lim_{x\to 0}\frac{x}{\sin x}=1$ ***(R1)***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

continued…

Question 9 continued

METHOD 3

expressing either $x\sin x$ or $1-\cos x$ as a Maclaurin series with at least two terms ***(M1)***

$$\lim_{x\to 0}\left(\frac{x\left(x-\frac{x^3}{3!}\left(+\frac{x^5}{5!}\ldots\right)\right)}{\frac{x^2}{2!}-\frac{x^4}{4!}\left(+\frac{x^6}{6!}+\ldots\right)}\right)$$ ***A1***

recognizing numerator and denominator are multiples of x^2 ***(M1)***

$$\lim_{x\to 0}\left(\frac{1-\frac{x^2}{3!}\left(+\frac{x^4}{5!}-\ldots\right)}{\frac{1}{2!}-\frac{x^2}{4!}\left(+\frac{x^4}{6!}+\ldots\right)}\right)$$ ***A1***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

METHOD 4

Expressing in terms of half angles ***M1***

$$\lim_{x\to 0}\frac{2x\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$ ***A1A1***

$$=\lim_{x\to 0}\frac{x\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=2\lim_{x\to 0}\frac{\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}$$ ***(M1)***

Using $\lim_{x\to 0}\frac{x}{\tan x}=1$ ***M1***

=2 ***A1***

[6 marks]

Section B

10. (a) (i) **METHOD 1**

attempt to equate differences of consecutive terms ***(M1)***

$(3-2k)-(k-5)=(5k+3)-(3-2k)$ **OR** $(k-5)-(3-2k)=(3-2k)-(5k+3)$ ***A1***

$8-3k=7k$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 2 (system of equations)

<u>**TWO**</u> correct equations involving k and d ***A1***

$k-5+d=3-2k$ OR $3-2k+d=5k+3$ OR $k-5+2d=5k+3$

OR $\frac{3}{2}\left(2(k-5)+2d\right)=k-5+3-2k+5k+3$ (or equivalent)

valid attempt to solve their system of equations using substitution or elimination ***(M1)***

$(d=5.6)$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 3 (in terms of k)

$\frac{3}{2}\left(k-5+5k+3\right)=k-5+3-2k+5k+3$ (or equivalent) ***A1***

combining like terms ***(M1)***

$9k-3=4k+1$ **OR** $5k=4$ (or equivalent)

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 4 (arithmetic mean)

attempt to find mean of u_1 and u_3 ***(M1)***

$\frac{(k-5)+(5k+3)}{2}=3-2k$ ***A1***

$3k-1=3-2k$

$(k=)\ \frac{4}{5}$ ***A1***

(ii) substituting their value of k into expression for u_3 ***(A1)***

$(u_3=)\ 5\times\frac{4}{5}+3$

$=7$ ***A1***

[5 marks]

continued…

Question 10 continued

(b) (i) substituting $k = 12$ into u_1, u_2 or u_3 ***(M1)***

$(u_1 =)\ 7$ **AND** $(u_2 =) -21$ **AND** $(u_3 =)\ 63$ ***(A1)***

$(r =)\ \frac{-21}{7} = \frac{63}{-21}\ (= -3)$ **OR** $(-21)^2 = 7 \times 63$ **OR** $r = -3$ ***R1***

u_1, u_2 and u_3 are in geometric sequence ***AG***

(ii) since $|r| \geq 1$ (accept $|r| > 1$ or $r < -1$) ***R1***

hence not convergent ***AG***

[4 marks]

(c) (i) attempts to find ratios, in terms of k, of consecutive terms and equating ***(M1)***

$\frac{(3-2k)}{(k-5)} = \frac{(5k+3)}{(3-2k)}$ **OR** $(3-2k)^2 = (k-5)(5k+3)$ (or equivalent)

$9 - 12k + 4k^2 = 5k^2 - 22k - 15$ ***A1***

Note: Award ***A1*** for correct expansion of all brackets leading to given result.

$k^2 - 10k - 24 = 0$ ***AG***

(ii) recognizing need to factorize, complete the square or substitute into quadratic formula ***(M1)***

$(k+2)(k-12)\ (= 0)$ **OR** $(k-5)^2 - 49\ (= 0)$ **OR** $k = \frac{10 \pm \sqrt{196}}{2}$

$k = -2$ (accept $k = -2$ **and** $k = 12$) ***A1***

substituting their value of k (other than $k = 12$) to find u_1, u_2 or u_3 ***(M1)***

$(u_1 =) -7$ **AND** $(u_2 =)\ 7$ **AND** $(u_3 =) -7$ ***A1***

(iii) $(S_{2m} =)\ 0$ ***A1***

[7 marks]

Total [16 marks]

11. (a) $\vec{BC} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ **OR** $\vec{BA} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$ ***(A1)***

$(3, 0, 2)$ ***A1***

[2 marks]

(b) **EITHER**

recognizing to find midpoint of $[AC]$ or $[BD]$ ***(M1)***

$\left(\frac{-1+1}{2}, \frac{-2-4}{2}, \frac{4-0}{2}\right)$ **OR** $\left(\frac{-3+3}{2}, \frac{-6+0}{2}, \frac{2+2}{2}\right)$

OR

Let $E(a, b, c)$

$\vec{AE} = \vec{EC} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 4-c \end{pmatrix}$ **OR** $\vec{AE} = \frac{1}{2}\vec{AC} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$ ***(M1)***

THEN

$(0, -3, 2)$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) (i) $\vec{AB} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}$ **AND** $\vec{AD} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ ***A1***

Note: Award ***A1*** for finding both vectors, one of which may have been seen in (a).

$$\begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \\ -12 \end{pmatrix}$$

$m = 12$ ***A1***

(ii) $\sqrt{(-12)^2 + 12^2 + (-12)^2}$ **OR** $12\sqrt{(-1)^2 + 1^2 + (-1)^2}$ ***(A1)***

$= 12\sqrt{3}$ ***A1***

[4 marks]

(d) substituting cross product and a point into the equation of a plane ***(M1)***

$-x + y - z = -5$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) let a normal to Π_1 be $\boldsymbol{n}_1$ and a normal to Π_2 be $\boldsymbol{n}_2$

Eg. $\boldsymbol{n}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$

recognizing angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is required ***(M1)***

$|\boldsymbol{n}_2| = \sqrt{75}$ ***A1***

$\cos\theta = \frac{-1\times 5 + 1\times 1 + (-1)\times(-7)}{\sqrt{3}\times\sqrt{75}} \left(= \frac{3}{\sqrt{3}\times\sqrt{75}} \right)$ ***A1***

> **Note:** The negative of this may be found but to obtain the final ***A1*** there needs to be a recognition that this obtains the obtuse angle for the minus to be dropped.

$\cos\theta = \frac{1}{5}$ ***AG***

[3 marks]

(f) attempt to substitute normal to Π_1 and their E into vector equation of line ***(M1)***

$(\boldsymbol{r} =) \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ ***A1***

substitutes for λ into Π_2 ***(M1)***

$5(-\lambda) + (-3+\lambda) - 7(2-\lambda) = 1$

$\lambda = 6$ ***A1***

$(-6, 3, -4)$ ***A1***

[5 marks]

Total [18 marks]

12. (a) substituting into modulus formula with either correct real part or correct imaginary part ***(M1)***

$\sqrt{(x-2)^2+(y-1)^2} \ (=3)$ ***(A1)***

$x^2-4x+4+y^2-2y+1=9$ ***A1***

$x^2+y^2-4x-2y-4=0$ ***AG***

[3 marks]

(b) **METHOD 1**

recognizing the need to express $\frac{z+p}{z-1}$ in the form $a+\mathrm{i}b$ ***(M1)***

$\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ **OR** $\frac{z+p}{z-1}\times\frac{(x-y\mathrm{i})-1}{(x-y\mathrm{i})-1}$ **OR** $\frac{(x+p)+\mathrm{i}y}{(x-1)+\mathrm{i}y}\times\frac{(x-1)-\mathrm{i}y}{(x-1)-\mathrm{i}y}$ ***A1***

expanding their $\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ ***(M1)***

$\frac{zz^*-z+pz^*-p}{zz^*-z-z^*+1}$ **OR** $\frac{x^2+y^2-x-y\mathrm{i}+px-py\mathrm{i}-p}{x^2+y^2-x-y\mathrm{i}-x+y\mathrm{i}+1}$

OR $\frac{(x+p)(x-1)+y^2+((x-1)y-(x+p)y)\mathrm{i}}{(x-1)^2+y^2}$

$\frac{x^2+y^2+(p-1)x-p-(p+1)y\mathrm{i}}{x^2+y^2-2x+1}$ (or equivalent) (seen anywhere) ***A1***

$\left(\Rightarrow \arg\left(\frac{z+p}{z-1}\right)=\right) \arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)$ ***(A1)***

equating their $\arg\left(\frac{z+p}{z-1}\right)$ to $\frac{\pi}{4}$ **OR** their expression for $\frac{z+p}{z-1}$ in x and y to 1

OR equating real and imaginary parts ***(M1)***

$\arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)=\frac{\pi}{4}$ **OR** $\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}=1$

$-(p+1)y=x^2+y^2+(p-1)x-p$ ***A1***

$x^2+y^2+(p-1)x+(p+1)y-p=0$ ***AG***

continued…

Question 12 continued

METHOD 2

$\arg\left(\frac{z+p}{z-1}\right) = \arg(z+p) - \arg(z-1)$ ***(M1)***

$= \arg(x+p+y\mathrm{i}) - \arg(x-1+y\mathrm{i})$ ***A1***

$= \arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)$ ***A1***

attempt to use compound angle formula for tan ***(M1)***

$$\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right) = \frac{\tan\left(\arctan\left(\frac{y}{x+p}\right)\right) - \tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}{1+\tan\left(\arctan\left(\frac{y}{x+p}\right)\right)\times\tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}$$

$$= \frac{\frac{y}{x+p} - \frac{y}{x-1}}{1+\frac{y}{x+p}\times\frac{y}{x-1}}$$ ***A1***

equating their expression for $\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right)$ to $\tan\frac{\pi}{4}$ ***(M1)***

$$\frac{\frac{y}{x+p} - \frac{y}{x-1}}{1+\frac{y}{x+p}\times\frac{y}{x-1}} = \tan\frac{\pi}{4} \quad \left(\Rightarrow \frac{(x-1)y-(x+p)y}{(x+p)(x-1)+y^2} = 1\right)$$

$(x+p)(x-1)+y^2 = -y-py$ ***A1***

$x^2 - x + px - p + y^2 = -y - py$

$x^2 + y^2 + (p-1)x + (p+1)y - p = 0$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) $\left(\arg\left(\frac{z+4}{z-1}\right)=\frac{\pi}{4}\Rightarrow\right)$ $x^2+y^2+3x+5y-4=0$ (seen anywhere) ***A1***

solving simultaneously their equations in x and y ***(M1)***

$7x+7y=0$ $(\Rightarrow x=-y)$ (or equivalent) ***A1***

substituting to obtain an equation in either x or y ***(M1)***

$x^2-x-2=0$ **OR** $y^2+y-2=0$

solving for either x or y ***(M1)***

$x=-1$ and $x=2$ **OR** $y=-2$ and $y=1$ ***A1***

$(z_1=)$ $-1+\mathrm{i}$ **AND** $(z_2=)$ $2-2\mathrm{i}$ (or equivalent) ***A1***

Note: Accept $x=-1,\ y=1$ **AND** $x=2,\ y=-2$.

[7 marks]

(d) let z_1, z_2, z_3 and z_4 be roots of $z^4+az^3+bz^2+cz+d=0$

EITHER

$a=-(z_1+z_2+z_3+z_4)$ ***(A1)***

OR

expanding $(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ to obtain term in z^3

$-(z_1+z_2)z^3-(z_3+z_4)z^3$ ***(A1)***

THEN

recognizing that z_3 and z_4 are conjugates of z_1 and z_2 (seen anywhere) ***(M1)***

roots are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $2-2\mathrm{i}$, $2+2\mathrm{i}$ ***A1***

$a=-((-1+\mathrm{i})+(-1-\mathrm{i})+(2-2\mathrm{i})+(2+2\mathrm{i}))$

$=-2$ ***A1***

[4 marks]

Total [21 marks]

8824 – 9700M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2024

Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

SECTION A

1. (a) $\frac{1}{2}r^2\theta = 48$ OR $\frac{1}{2}r^2(1.5) = 48$ ***(A1)***

attempt to solve their equation to find r or r^2 ***(M1)***

Note: To award the ***M1***, candidate's equation must include r^2 and $\theta = 1.5$, and they must attempt to isolate r^2 or r.

$r^2 = 64$

$r = 8$ (cm) ***A1***

[3 marks]

(b) evidence of summing the two radii and the arc length ***(M1)***

perimeter $= 2r + r\theta$

$= 16 + 8(1.5)$

$= 28$ (cm) ***A1***

[2 marks]

Total [5 marks]

2. (a) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

$\mathrm{P}(A \cap B) = 0.65 + 0.45 - 0.85$ (or equivalent) ***(A1)***

$= 0.25$ ***A1***

[3 marks]

(b) $\mathrm{P}(A' \cap B') = 0.15$ (may be seen in Venn diagram) ***(A1)***

attempt to substitute their values into $\mathrm{P}(A' \mid B') = \frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')}$ ***(M1)***

$\mathrm{P}(A' \mid B') = \frac{0.15}{0.55}$

$= \frac{15}{55}\left(= \frac{3}{11}\right)$ ***A1***

[3 marks]

Total [6 marks]

3. **METHOD 1**

attempt to expand $(3n+2)^2-(3n-2)^2$ ***M1***

Note: Award ***M0*** for invalid attempts such as $(3n+2)^2=9n^2+4$.

$=9n^2+12n+4-(9n^2-12n+4)$ or equivalent ***A1***

$=24n$ OR $12n+12n$ ***A1***

$=12(2n)$ OR $\frac{24}{12}=2$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent) ***R1***

so is a multiple of 12 ***AG***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

METHOD 2

use of $a^2-b^2=(a+b)(a-b)$ where $a=3n+2,\, b=3n-2$ ***M1***

$=(3n+2+3n-2)(3n+2-3n+2)$

$=6n\times 4$ ***A1***

$=24n$ ***A1***

$=12(2n)$ OR $\frac{24n}{12}=2n$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent)

R1

so is a multiple of 12 ***AG***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

continued…

Question 3 continued.

METHOD 3

base case $n=1$: $\left(3(1)+2\right)^2-\left(3(1)-2\right)^2=25-1=24$

so true for $n=1$ ***A1***

assume true for $n=k$ i.e. $(3k+2)^2-(3k-2)^2$ is a multiple of 12 ***M1***

consider $n=k+1$:

$\left(3(k+1)+2\right)^2-\left(3(k+1)-2\right)^2$

$\left((3k+2)+3\right)^2-\left((3k-2)+3\right)^2$

$(3k+2)^2+6(3k+2)+9-\left((3k-2)^2+6(3k-2)+9\right)$

$(3k+2)^2-(3k-2)^2+24$

using the assumption $(3k+2)^2-(3k-2)^2=12M$

$12M+24$

$12(M+2)$

which is a multiple of 12, hence true for $n=k+1$ ***A1***

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

[4 marks]

4. attempt to substitute into cosine rule ***(M1)***

$\left(\cos 2\theta =\right)\dfrac{4^2+6^2-5^2}{2\times 4\times 6}$ OR $5^2=4^2+6^2-2\times 4\times 6\times \cos 2\theta$

$\left(\cos 2\theta =\right)\dfrac{27}{48}\left(=\dfrac{9}{16}\right)$ ***(A1)***

attempt to use $\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$\cos^2\theta = \dfrac{1+\frac{27}{48}}{2}\quad\left(=\dfrac{1+\frac{9}{16}}{2}\right)$

$\cos^2\theta = \dfrac{75}{96}\quad\left(=\dfrac{25}{32}\right)$ ***A1***

$\cos\theta = \left(\pm\right)\sqrt{\dfrac{75}{96}}\quad\left(=\sqrt{\dfrac{25}{32}}=\dfrac{5}{\sqrt{32}}\right)$ ***(A1)***

$=\dfrac{5}{4\sqrt{2}}$

$=\dfrac{5\sqrt{2}}{8}\quad\left(p=5, q=8\right)$ ***A1***

Note: The final answer must be positive.

[6 marks]

5. attempt to use $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ or $S_n = \frac{n}{2}\left[u_1 + u_n\right]$ to

set up at least one equation in u_1 and d ***(M1)***

$16 = u_1 + 9d$ and $100 = \frac{25}{2}\left[2u_1 + 24d\right]$ ***(A1)***

attempt to solve their two linear equations in u_1 and d simultaneously (must eliminate one variable) ***(M1)***

$d = -4 \left(\Rightarrow u_1 = 52\right)$ ***A1***

attempt to solve $u_k = 0$ with their d (or with their d and u_1) ***(M1)***

$\Rightarrow k = 14$ ***A1***

[6 marks]

6. (a) attempt to find critical values ***(M1)***

$x = \frac{3}{2}, x = 6$ ***(A1)***

$\frac{3}{2} < x < 6$ ***A1***

Note: Allow equivalent, and/or interval notation.

[3 marks]

(b) $k = \frac{3}{2}$ ***A1***

since we require $2x^2 - 15x + 18 \geq 0$ (and f must be one to one) ***R1***

OR

Does not obey horizontal line test for $x \geq \frac{3}{2}$ ***R1***

[2 marks]

Total [5 marks]

7. (a) attempt to find $f\left(0\right)=\sqrt{2},\ f\left(\frac{\pi}{4}\right)=1$ or $f\left(\frac{\pi}{2}\right)=\sqrt{2}$ or sketch of graph ***(M1)***

$1\le f\left(x\right)\le\sqrt{2}$ ***A1A1***

Note: Award ***A1A0*** for strong inequality seen. Allow equivalent, and/or interval notation.

[3 marks]

(b) consider $\pi\int_0^{\frac{\pi}{2}}\sec^2\left(x-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1***

Note: For the ***M1,*** condone incorrect or missing limits and omission of π.

$=\pi\left[\tan\left(x-\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$ ***A1***

$=\pi\left[\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)\right]$ ***(A1)***

$=\pi\left(1-\left(-1\right)\right)$

$=2\pi$ ***A1***

[4 marks]

Total [7 marks]

8. (a) $\vec{OP} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$ ***A1***

$= (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}$ ***AG***

[1 mark]

(b) **METHOD 1**

recognition that $\vec{OP} \cdot \vec{AB} = 0$ (may be seen anywhere) ***(M1)***

$[(1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}].[\boldsymbol{b} - \boldsymbol{a}](=0)$ ***A1***

attempt to multiply out scalar product ***M1***

$(1-\lambda)\boldsymbol{a}.\boldsymbol{b} + \lambda\boldsymbol{b}.\boldsymbol{b} - (1-\lambda)\boldsymbol{a}.\boldsymbol{a} - \lambda\boldsymbol{b}.\boldsymbol{a}(=0)$ ***(A1)***

attempt to substitute for $\boldsymbol{a}.\boldsymbol{b}$ and $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ ***(M1)***

$\frac{1}{4}(1-\lambda) + 4\lambda - (1-\lambda) - \frac{\lambda}{4}(=0)$ ***(A1)***

$1 - \lambda + 16\lambda - 4 + 4\lambda - \lambda = 0$

$18\lambda - 3 = 0$

$\lambda = \frac{1}{6}$ ***A1***

continued…

Question 8 continued.

METHOD 2

$\cos AOB = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{1}{8}$ ***A1***

attempt to use cosine rule to find AB ***M1***

$|AB|^2 = 1^2 + 2^2 - 2(1)(2)\left(\dfrac{1}{8}\right)$

$AB = \dfrac{3\sqrt{2}}{2}$ ***A1***

attempt to apply Pythagoras' Theorem twice: ***M1***

$|OP|^2 + \left(\dfrac{3\sqrt{2}}{2}\lambda\right)^2 = 1$ and

$|OP|^2 + \left(\dfrac{3\sqrt{2}}{2}(1-\lambda)\right)^2 = 4$ ***A1***

attempt to solve simultaneously: ***M1***

$\dfrac{9}{2}(1-\lambda)^2 - \dfrac{9}{2}\lambda^2 = 3$

$\lambda = \dfrac{1}{6}$ ***A1***

[7 marks]

Total [8 marks]

9. (a) **METHOD 1**

attempt to use identity $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$\tan\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\tan\theta - 1}{1+\tan\theta}$ ***A1***

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$\dfrac{\dfrac{\sin\theta}{\cos\theta} - 1}{1+\dfrac{\sin\theta}{\cos\theta}}$ OR $\dfrac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$

multiply through by the conjugate of the denominator $\dfrac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$ ***M1***

$= \dfrac{-\sin^2\theta - \cos^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \dfrac{-\left(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta\right)}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{-(1-\sin 2\theta)}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \dfrac{\sin 2\theta - 1}{\cos 2\theta}$ ***AG***

continued…

Question 9 continued.

METHOD 2

attempt to write $\tan\left(\theta-\frac{\pi}{4}\right)$ in terms of sin and cos: ***M1***

$$\left(\tan\left(\theta-\frac{\pi}{4}\right)\right)=\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta-\frac{\pi}{4}\right)}$$

attempt to use both sin and cos addition formulae: ***M1***

$$=\frac{\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}}{\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}}$$

$$=\frac{\sin\theta-\cos\theta}{\cos\theta+\sin\theta}$$ ***A1***

multiply through by the conjugate of the denominator $\frac{\cos\theta-\sin\theta}{\cos\theta-\sin\theta}$ ***M1***

$$=\frac{-\sin^2\theta-\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$ ***A1***

$$=\frac{-\left(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta\right)}{\cos^2\theta-\sin^2\theta}$$

$$=\frac{-\left(1-\sin 2\theta\right)}{\cos^2\theta-\sin^2\theta}$$ ***A1***

$$=\frac{\sin 2\theta-1}{\cos 2\theta}$$ ***AG***

continued…

Question 9 continued.

METHOD 3

attempt to use given identity $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$\tan\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\tan\theta - 1}{1+\tan\theta}$ ***A1***

multiply through by the conjugate of the denominator $\dfrac{1-\tan\theta}{1-\tan\theta}$ ***M1***

$\dfrac{(\tan\theta - 1)}{(1+\tan\theta)} \cdot \dfrac{(1-\tan\theta)}{(1-\tan\theta)}$ ***A1***

$\dfrac{2\tan\theta - \tan^2\theta - 1}{1-\tan^2\theta}\left(= \dfrac{2\tan\theta - \sec^2\theta}{1-\tan^2\theta}\right)$

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$= \dfrac{2\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{\cos^2\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{2\sin\theta\cos\theta - 1}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \dfrac{\sin 2\theta - 1}{\cos 2\theta}$ ***AG***

[6 marks]

continued…

Question 9 continued.

(b) recognition that $x = 2\theta \Rightarrow \theta = \frac{x}{2}$ ***M1***

$\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) = \sqrt{3}$

$\frac{x}{2} - \frac{\pi}{4} = \frac{\pi}{3}, \left(\frac{4\pi}{3}\right)$ ***A1***

$x = \frac{7\pi}{6}$ ***A1***

Note: Award ***A0*** if extra solutions outside the domain are seen.

[3 marks]

Total [9 marks]

SECTION B

10. (a) outer curved surface area is $2\pi(4r)h$ AND inner curved surface area is $2\pi rh$ ***(A1)***

area of each base (top and bottom) is $\pi(4r)^2 - \pi r^2$ ***(A1)***

$S = 2\left[\pi(4r)^2 - \pi r^2\right] + 2\pi(4r)h + 2\pi rh$ ***A1***

$= 30\pi r^2 + 10\pi rh$ ***AG***

[3 marks]

(b) $30\pi r^2 + 10\pi rh = 240\pi$

attempt to solve their equation for h or rh in terms of r (must isolate h or rh) ***(M1)***

$h = \dfrac{240 - 30r^2}{10r}\left(= \dfrac{24 - 3r^2}{r}\right)$ OR $rh = \dfrac{240 - 30r^2}{10}\left(= 24 - 3r^2\right)$ (or equivalent) ***A1***

uses volume = large cylinder – small cylinder ***(M1)***

$V = \pi(4r)^2 h - \pi r^2 h \quad \left(= 16\pi r^2 h - \pi r^2 h = 15\pi r^2 h\right)$ ***A1***

attempt to substitute in for h or rh ***(M1)***

$V = 15\pi r^2\left(\dfrac{24 - 3r^2}{r}\right)$ OR $V = 15\pi r\left(\dfrac{240 - 30r^2}{10}\right)\left(= 15\pi r\left(24 - 3r^2\right)\right)$ OR

$384\pi r - 48\pi r^3 - 24\pi r + 3\pi r^3$ ***A1***

$= 360\pi r - 45\pi r^3$ ***AG***

[6 marks]

(c) $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 360\pi - 135\pi r^2$ ***A1A1***

[2 marks]

continued…

Question 10 continued.

(d) **METHOD 1** (working with r)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi r^2=0$

$r^2=\frac{360}{135}\left(=\frac{8}{3}\right)$

$r=\sqrt{\frac{360}{135}}\left(=\sqrt{\frac{8}{3}}\right)$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

METHOD 2 (working with $p\sqrt{\frac{2}{3}}$)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi\left(p\sqrt{\frac{2}{3}}\right)^2=0$

$360-90p^2=0$

$p^2=4$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

[3 marks]

continued…

Question 10 continued.

(e) attempt to substitute their value of r into $V = 360\pi r - 45\pi r^3$ ***M1***

$$V = 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times \left(2\sqrt{\frac{2}{3}}\right)^3$$

$$= 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times 8 \times \frac{2}{3} \times \sqrt{\frac{2}{3}}$$

$$\left(= 720\pi\sqrt{\frac{2}{3}} - 240\pi\sqrt{\frac{2}{3}}\right)$$ ***(A1)***

$$= 480\pi\sqrt{\frac{2}{3}}$$ ***A1***

[3 marks]

Total [17 marks]

11. (a) **EITHER**

attempt to use l'Hôpital's ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\frac{2e^{2x}}{2e^{2x}}$$ ***A1***

OR

attempt to divide each term by e^{2x} ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)$$ ***A1***

THEN

$=1$ ***AG***

[2 marks]

(b) (i) attempt to use quotient rule or product rule ***M1***

$$\frac{dy}{dx}=\frac{\left(e^{2x}+1\right)2e^{2x}-\left(e^{2x}-1\right)2e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for second term in numerator. If denominator is incorrect award ***A1A0.***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

(ii) attempt to substitute for y and express as a single fraction ***M1***

$$1-y^2=1-\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}$$

$$=\frac{\left(e^{2x}+1\right)^2-\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}$$ or equivalent ***(A1)***

$$=\frac{e^{4x}+2e^{2x}+1-\left(e^{4x}-2e^{2x}+1\right)}{\left(e^{2x}+1\right)^2}\quad\text{OR}\quad\frac{2e^{2x}\times 2}{\left(e^{2x}+1\right)^2}$$ ***A1***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

[6 marks]

continued…

Question 11 continued.

(c) (i) attempt to use implicit differentiation ***M1***

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\right)-2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$=-2y\left(1-y^2\right)$ ***A1***

$=2y^3-2y$ ***AG***

(ii) $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\right)\left(6y^2-2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$=\left(6y^2-2\right)\left(1-y^2\right)\left(=8y^2-6y^4-2\right)$ ***A1***

[5 marks]

(d) attempt to evaluate at least three of y, y', y'', y''' at $x=0$ ***(M1)***

$y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y'''(0)=-2$ ***A1***

attempt to use Maclaurin series $y \approx y(0)+y'(0)x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\ldots$ ***M1***

$y \approx 0+1x+0x^2+\frac{(-2)}{3!}x^3+\ldots$

$y \approx x-\frac{x^3}{3}+\ldots$ ***A1***

[4 marks]

Total [17 marks]

12. (a) $|16\mathrm{i}| = 16$ and $\arg(16\mathrm{i}) = \dfrac{\pi}{2}$ ***(A1)***

attempt to use De Moivre's Theorem ***(M1)***

$z_1 = 2\left(\cos\left(\dfrac{\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{8}\right)\right)$ ***A1***

attempts to find other solutions using $z = 2\left(\cos\left(\dfrac{\pi}{8} + \dfrac{\pi k}{2}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{8} + \dfrac{\pi k}{2}\right)\right)$ or equivalent ***(M1)***

$z_2 = 2\left(\cos\left(\dfrac{5\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{5\pi}{8}\right)\right)$ (or any other root) ***A1***

$z_3 = 2\left(\cos\left(\dfrac{9\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{9\pi}{8}\right)\right)$ and $z_4 = 2\left(\cos\left(\dfrac{13\pi}{8}\right) + \mathrm{i}\sin\left(\dfrac{13\pi}{8}\right)\right)$ ***A1***

Note: Award a maximum of ***(A1)(M1)A1(M1)A1A0*** for more than four roots or any roots outside the range.

Note: Allow use of r-cis form throughout.

[6 marks]

(b) attempt to evaluate a ratio with their roots eg $\dfrac{z_2}{z_1}$ ***(M1)***

$$\frac{z_2}{z_1} = \frac{2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)}{2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)}$$ or equivalent

$= \left(\cos\left(\dfrac{\pi}{2}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{2}\right)\right)$ ***(A1)***

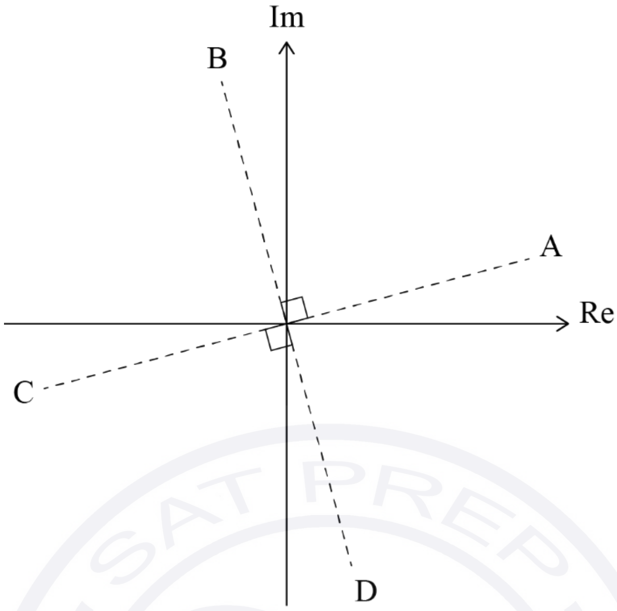
$= \mathrm{i}$ ***A1***

[3 marks]

continued…

Question 12 continued.

(c)



point A in approximately correct place in first quadrant ***A1***

points A, B, C and D approximately the same distance from the origin ***A1***

approximate angular separation of $\frac{\pi}{2}$ ***A1***

Note: Dotted lines not required.

[3 marks]

continued…

Question 12 continued.

(d) **EITHER**

$\left(z_i^{\ *}\right)^4 = \left(z_i^{\ 4}\right)^*$ ***(A1)***

$= (16\text{i})^*$ ***(A1)***

OR

$z_1^* = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$ ***(A1)***

$\left(z_1^*\right)^4 = 2^4\left(\cos\left(-\frac{4\pi}{8}\right) + \text{i}\sin\left(-\frac{4\pi}{8}\right)\right)$ ***(A1)***

OR

$z_1 z_2 z_3 z_4 = -16\text{i}$ ***(A1)***

$\left(z_1 z_2 z_3 z_4\right)^* = (-16\text{i})^*$

$z_1^{\ *} z_2^{\ *} z_3^{\ *} z_4^{\ *} = 16\text{i}$ ***(A1)***

THEN

$\left(z^4 =\right) -16\text{i}$ ***A1***

$(a = 0,\ b = -16)$

[3 marks]

(e) $\arg w_1 \left(= \frac{\pi}{8} + \frac{\pi}{4}\right) = \frac{3\pi}{8}$ ***A1***

$\text{AB} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \text{AA}' = \sqrt{2}$

$\Rightarrow \text{OA}' = |w_1| = \sqrt{2}$ ***A1***

$\therefore w_1 = \sqrt{2}\,\text{cis}\,\frac{3\pi}{8}$

considers when $\arg\left(w_1^{\ p}\right) \in \mathbb{Z}^+$ (multiple of 2π) ***(M1)***

$\Rightarrow p = 16$ ***A1***

$\Rightarrow q = 8$ ***A1***

[5 marks]

Total [20 marks]

2224 – 7101M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 1

33 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form equation for the sum of frequencies$=16$ or mean$=3$ ***(M1)***

$p+q+4+2+3=16\left(\Rightarrow p+q=7\right)$ ***A1***

$\frac{p+2q+12+8+18}{16}=3\left(\Rightarrow p+2q=10\right)$ OR $\frac{p+2q+12+8+18}{9+p+q}=3\left(\Rightarrow 2p+q=11\right)$ ***A1***

attempt to eliminate one variable from their equations ***(M1)***

$p+2\left(7-p\right)+38=48$ OR $2\left(7-q\right)+q=11$

$p=4$ and $q=3$ ***A1***

Note: Award ***M1A0A0M0A1*** for $p=4, q=3$ with no working.

[5 marks]

(b) mean final score $=30$ ***A1***

[1 mark]

Total [6 marks]

2. (a) $\log_{10}1-\log_{10}a$ OR $\log_{10}a^{-1}=-\log_{10}a$ OR $\log_{10}10^{-\frac{1}{3}}$ OR $10^x=\dfrac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$=-\dfrac{1}{3}$ ***A1***

[2 marks]

(b) $\dfrac{\log_{10}a}{\log_{10}1000}$ OR $\dfrac{1}{3}\log_{1000}10$ OR $\log_{1000}\sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}}=1000^x\left(=\left(10^3\right)^x\right)$ ***(A1)***

$\dfrac{\log_{10}a}{3}$ OR $\dfrac{1}{3}\log_{1000}1000^{\frac{1}{3}}$ OR $\log_{1000}1000^{\frac{1}{9}}$ OR $3x=\dfrac{1}{3}$ ***(A1)***

$=\dfrac{1}{9}$ ***A1***

[3 marks]

Total [5 marks]

3. (a) $2r + r\theta = 10$ ***A1***

$\frac{1}{2}r^2\theta = 6.25$ ***A1***

attempt to eliminate θ to obtain an equation in r ***M1***

correct intermediate equation in r ***A1***

$10 - 2r = \frac{25}{2r}$ OR $\frac{10}{r} - 2 = \frac{25}{2r^2}$ OR $\frac{1}{2}r^2\left(\frac{10}{r} - 2\right) = 6.25$ OR $12.5 + 2r^2 = 10r$

$4r^2 - 20r + 25 = 0$ ***AG***

[4 marks]

(b) attempt to solve quadratic by factorizing or use of formula or completing the square ***(M1)***

$(2r-5)^2 = 0$ OR $r = \frac{20 \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}\left(= \frac{20 \pm \sqrt{400-400}}{8}\right)$

$r = \frac{5}{2}$ ***A1***

attempt to substitute their value of r into their perimeter or area equation ***(M1)***

$\theta = \frac{10 - 2\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}$ or $\theta = \frac{25}{2\left(\frac{5}{2}\right)^2}$

$\theta = 2$ ***A1***

[4 marks]

Total [8 marks]

4. (a) recognising $\cos x = 2\sin x\cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x - coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award ***(M1)(A1)A0*** for solutions of 30° and 150°.

[3 marks]

(b) **METHOD 1**

attempt to integrate $\pm(\cos x - \sin 2x)$ ***(M1)***

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}}(\cos x - \sin 2x)\,dx$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}}(\cos x - 2\sin x\cos x)\,dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for $\pm$correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract ***M1***

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B. Accept work done in degrees.

continued…

Question 4 continued

METHOD 2

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x\mathrm{d}x = [\sin x]_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \quad \text{and} \quad \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x\mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right) \quad \text{and} \quad -\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right) \text{ OR } \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x\mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x\mathrm{d}x$$

$$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \quad \left(= -\frac{1}{4}\right)$$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B. Accept work done in degrees.

[4 marks]

Total [7 marks]

5. (a) $S_n = \dfrac{10^n - 1}{9}$ ***A1***

$(a = 10\,, b = 9)$

[1 mark]

(b) **METHOD 1**

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \dfrac{10-1}{9} + \dfrac{10^2-1}{9} + \ldots + \dfrac{10^n-1}{9}$ ***(A1)***

$= \dfrac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\dfrac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10+10^2+10^3+\ldots+10^n = \dfrac{10\left(10^n-1\right)}{10-1}$ and $-1-1-1\ldots=-n$ ***A1***

$= \dfrac{\dfrac{10\left(10^n-1\right)}{10-1}-n}{9}$ OR $\dfrac{9\left(\dfrac{10\left(10^n-1\right)}{10-1}\right)-9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \dfrac{10\left(10^n-1\right)-9n}{81}$ ***AG***

continued…

Question 5 continued

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$$\sum_{i=1}^{n}\frac{10^i-1}{9} \quad \left(=\frac{1}{9}\left(\sum_{i=1}^{n}10^i-\sum_{i=1}^{n}1\right)\right)$$

$$\sum_{i=1}^{n}10^i=\frac{10\left(10^n-1\right)}{9}$$ ***A1***

$$\sum_{i=1}^{n}1=n$$ ***A1***

$$=\frac{1}{9}\left(\frac{10\left(10^n-1\right)}{9}-n\right) \text{ OR } \frac{1}{9}\left(\frac{10\left(10^n-1\right)-9n}{9}\right)$$ ***A1***

$$=\frac{10\left(10^n-1\right)-9n}{81}$$ ***AG***

continued…

Question 5 continued

METHOD 3

let $\mathrm{P}(n)$ be the proposition that $S_1+S_2+S_3+...+S_n=\dfrac{10\left(10^n-1\right)-9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS}=S_1=\dfrac{10^1-1}{9}=1$ and $\mathrm{RHS}=\dfrac{10\left(10^1-1\right)-9(1)}{81}=1$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *i.e.* $S_1+S_2+S_3+...+S_k=\dfrac{10\left(10^k-1\right)-9k}{81}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$S_1+S_2+S_3+...+S_{k+1}=\dfrac{10\left(10^k-1\right)-9k}{81}+\dfrac{10^{k+1}-1}{9}$

$=\dfrac{10^{k+1}-10-9k+9\left(10^{k+1}\right)-9}{81}$ ***A1***

$=\dfrac{10\left(10^{k+1}-1\right)-9(k+1)}{81}$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true ***R1***

(for all integers $n\geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

[4 marks]

Total [5 marks]

6. METHOD 1

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1.***

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2 \mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin(u)\,\mathrm{d}u$

Note: Award ***M1*** for $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$=\left[-\frac{\pi}{2}\cos(u)\right]_0^{\frac{\pi}{4}}$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}} 2x\sin\left(x^2\right)\mathrm{d}x$ OR $\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}\sin\left(x^2\right)\mathrm{d}\left(x^2\right)$

$=\left[-\frac{\pi}{2}\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ ***A1***

Note: Condone incorrect or absent limits for ***M1***. The correct limits may be seen or implied by later work for the ***A1***.

THEN

$=\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right)-\left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4}+\frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}+1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$ ***A1***

$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

continued…

Question 6 continued

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$$\pi\int_0^{\frac{\sqrt{\pi}}{2}} \left(f(x)\right)^2 \mathrm{d}x$$

$$\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\sin\left(x^2\right)\mathrm{d}x$$ ***A1***

attempt to use integration by substitution ***M1***

$$u=\cos\left(x^2\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-2x\sin\left(x^2\right)$$

Note: Award ***M1*** for $u=\cos\left(x^2\right)$

$$=-\frac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1} \mathrm{d}u$$

$$=\left[-\frac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$$ (or equivalent) ***A1A1***

Note: Condone incorrect or absent limits for ***M1***. ***A1*** for $-\frac{\pi}{2}u$ and ***A1*** for both correct limits.

$$=\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}} \text{ OR } \frac{\pi}{2}-\frac{\pi\sqrt{2}}{4} \text{ OR } \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right) \text{ OR } \frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)$$ ***A1***

$$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$$ ***AG***

Total [6 marks]

7. Base case $n=1$: LHS $= {}^{1}C_1 = 1$ and RHS $= {}^{2}C_2 = 1$, so true for $n=1$ ***R1***

> **Note:** Award ***R0*** if the value of ${}^{1}C_1$ and ${}^{2}C_2$ are not evaluated.
> Subsequent marks can still be awarded.

assume true for $n=k$ ie $\sum_{r=1}^{k} {}^{r}C_1 = {}^{k+1}C_2$ for some $k \in \mathbb{Z}^+$ ***M1***

> **Note:** The assumption of truth must be clear.
> Award ***M0*** for statements such as "let $n=k$" or "$n=k$ is true".
> Subsequent marks can still be awarded.

consider $n=k+1$

LHS $= \sum_{r=1}^{k+1} {}^{r}C_1$

$= \sum_{r=1}^{k} {}^{r}C_1 + {}^{k+1}C_1$ ***(M1)***

$= {}^{k+1}C_2 + {}^{k+1}C_1$ OR $\dfrac{(k+1)!}{2(k-1)!} + \dfrac{(k+1)!}{k!}$ ***A1***

EITHER

attempt to cancel factorials and use a common denominator ***M1***

$= \dfrac{(k+1)k + 2(k+1)}{2} \left(= \dfrac{(k+2)(k+1)}{2} \right)$

OR

attempt to use a common denominator ***M1***

$= \dfrac{k(k+1)!}{2k!} + \dfrac{2(k+1)!}{2k!} \left(= \dfrac{(k+2)(k+1)!}{2k!} \right)$

THEN

$= \dfrac{(k+2)!}{2!k!} \quad \left(= \dfrac{(k+2)!}{2!(k+2-2)!} \right)$ ***A1***

$= {}^{k+2}C_2$

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n \in \mathbb{Z}^+$ ***R1***

> **Note:** Only award the final ***R1*** if 4 of the previous 6 marks have been awarded.

Total *[7 marks]*

Note: Throughout this question, condone presence of any additional terms once the first two correct terms are seen.

8. (a) (i) $\sin\left(x^2\right) = x^2 - \dfrac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \dfrac{x^6}{3!} + \ldots \left(= x^2 - \dfrac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

(ii) **METHOD 1**

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \dfrac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\dfrac{x^6}{3!}\right)^2 + \ldots$.

$= x^4 - \dfrac{2x^8}{3!} + \ldots \left(= x^4 - \dfrac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \dfrac{1 - \cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \dfrac{1}{2}\left(1 - \left(1 - \dfrac{\left(2x^2\right)^2}{2!} + \dfrac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \dfrac{8x^8}{4!} + \ldots \left(= x^4 - \dfrac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[5 marks]

continued…

Question 8 continued

(b) **METHOD 1**

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=\frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$=4x^3-\frac{8x^7}{3}+\dots$ ***A1***

METHOD 2

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=2x\sin\left(2x^2\right)$ ***(M1)***

$=2x\left(2x^2-\frac{\left(2x^2\right)^3}{3!}+\dots\right)$

$=4x^3-\frac{8x^7}{3}+\dots$ ***A1***

METHOD 3

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=4x\left(x^2-\frac{x^6}{3!}+\dots\right)\left(1-\frac{x^4}{2!}+\dots\right)$ ***(A1)***

$=4x^3-\frac{8x^7}{3}+\dots$ ***A1***

continued…

Question 8 continued

METHOD 4

recognition that $2x\cos\left(x^2\right)=\frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

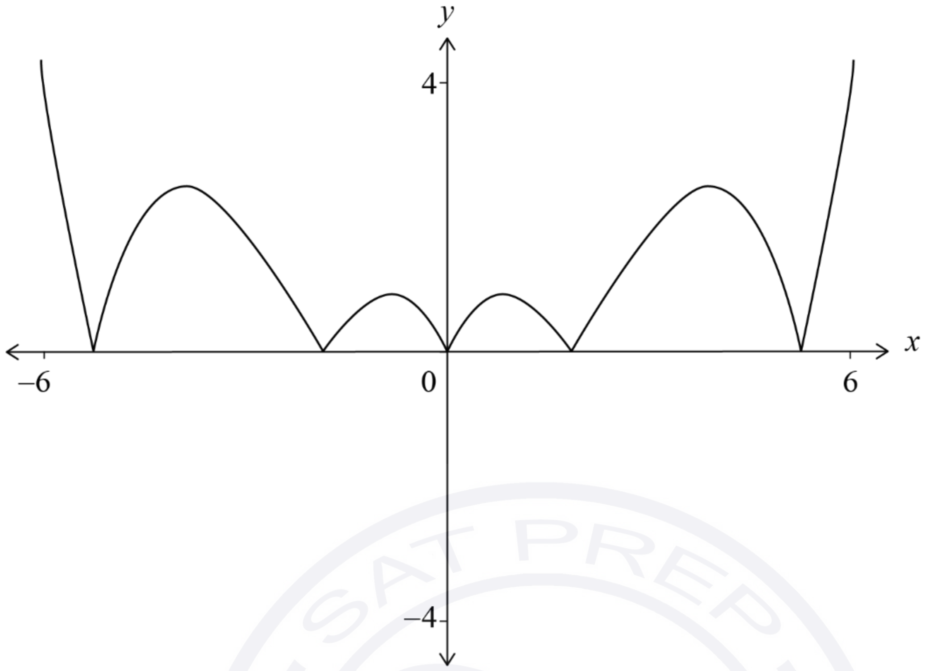
$=2\left(x^2-\frac{x^6}{3!}+\dots\right)\left(2x-\frac{6x^5}{2!}+\dots\right)$

$=4x^3-\frac{8x^7}{3}+\dots$ ***A1***

[2 marks]

Total [7 marks]

9. (a)



reflection of all negative sections in x-axis ***(M1)***

approximately correct graph with sharp points (cusps) at x-intercepts ***A1***

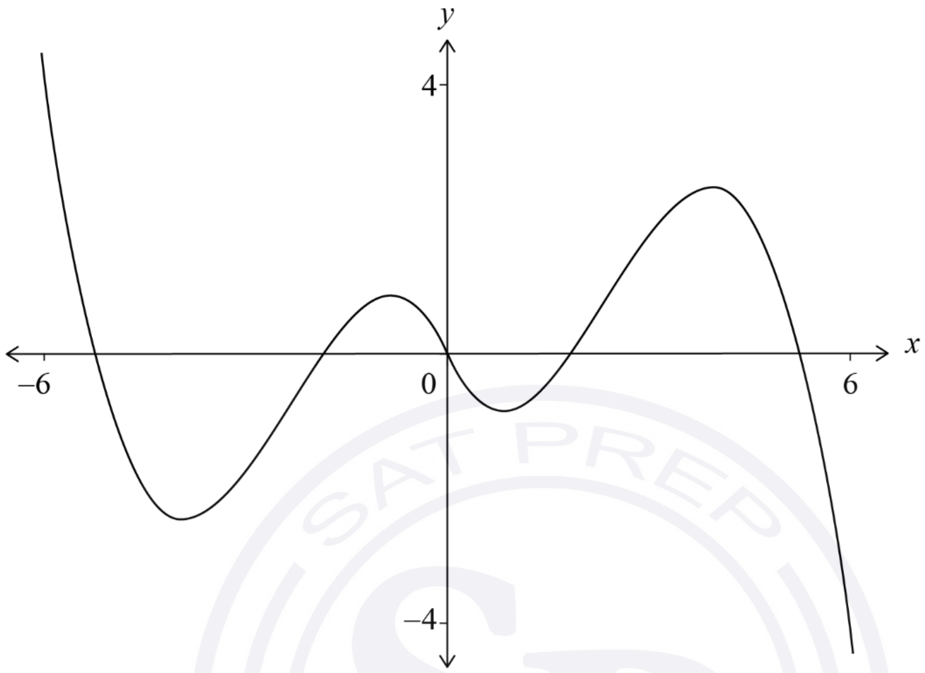
Note:	Award ***A1*** only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]

continued…

Question 9 continued

(b)



A1A1

Note: Award ***A1*** for right hand side unchanged and ***A1*** for rotation 180° about the origin.

[2 marks]

(c) (i) -1.6 ***A1***

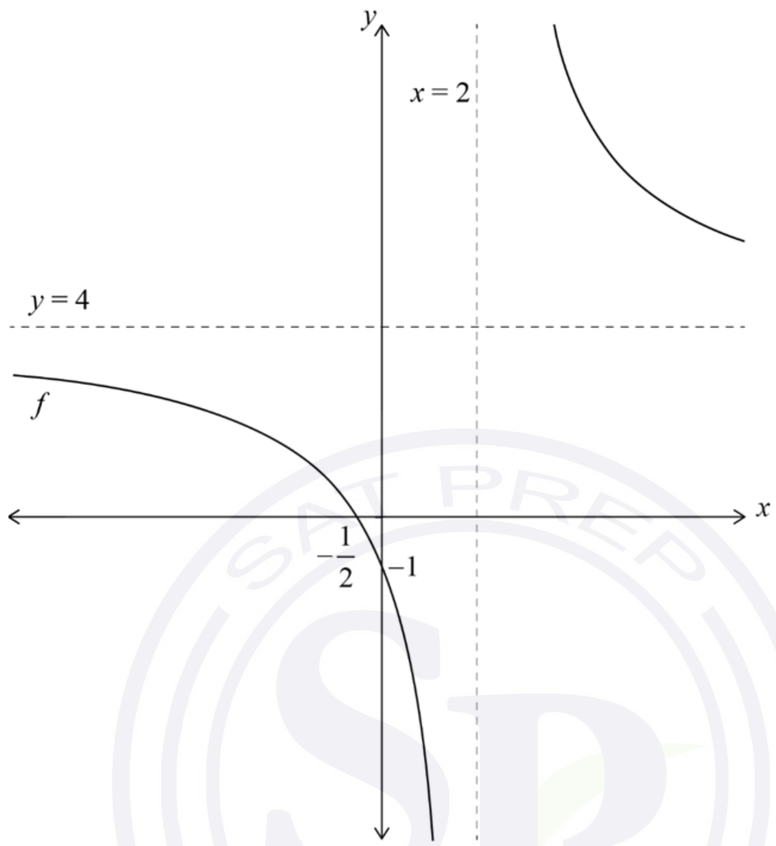
(ii) 3.2 ***A1***

[2 marks]

Total [6 marks]

Section B

10. (a)



vertical asymptote $x = 2$ sketched and labelled with correct equation *A1*

horizontal asymptote $y = 4$ sketched and labelled with correct equation *A1*

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x-axis, -1 on y-axis *A1A1*

two branches in correct opposite quadrants with correct asymptotic behaviour *A1*

Note:	These marks may be awarded independently.

[5 marks]

continued…

Question 10 continued

(b) $y \neq 4$ (or equivalent) ***A1***

[1 mark]

(c) $2+\frac{5}{2}$ OR $\left(-\frac{1}{2}\right)+2\times\frac{5}{2}$ OR $\frac{-\frac{1}{2}+p}{2}=2$ OR $-4=-p+\frac{1}{2}$ ***A1***

$p=\frac{9}{2}$ ***AG***

[1 mark]

(d) **METHOD 1**

attempt to substitute both roots to form a quadratic ***(M1)***

EITHER

$\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $x^2-\left(-\frac{1}{2}+\frac{9}{2}\right)x+\left(-\frac{1}{2}\times\frac{9}{2}\right)$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

OR

$(2x+1)(2x-9)=4\left(x^2-4x-\frac{9}{4}\right)$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

Note: Award ***A1*** for each correct value. They must be stated explicitly.

continued…

Question 10 continued

METHOD 2

$-\frac{b}{2}=2$ OR $4+b=0 \Rightarrow b=-4$ ***A1***

attempt to form a valid equation to find c using their b ***(M1)***

$\left(-\frac{1}{2}\right)^2+-4\left(-\frac{1}{2}\right)+c=0$ OR $\left(\frac{9}{2}\right)^2+-4\left(\frac{9}{2}\right)+c=0$

$c=-\frac{9}{4}$ ***A1***

METHOD 3

attempt to form two valid equations in b and c ***(M1)***

$\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)+c=0$, $\left(\frac{9}{2}\right)^2+b\left(\frac{9}{2}\right)+c=0$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2+k$ and substitute for x, h and $g(x)$ ***(M1)***

$\left(-\frac{1}{2}-2\right)^2+k=0 \Rightarrow k=-\frac{25}{4}$

$(x-2)^2-\frac{25}{4}$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

[3 marks]

continued…

Question 10 continued

(e) attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) ***(M1)***

$y = -\frac{25}{4}$ ***A1***

[2 marks]

(f) $\frac{4x+2}{x-2} = \left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $\frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$

attempt to form a cubic equation ***(M1)***

EITHER

$4x+2 = (x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $4x+2 = \left(x^2-4x-\frac{9}{4}\right)(x-2)$ OR

$(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)-4x-2$ OR $(x-2)\left(x^2-4x-\frac{9}{4}\right)-4x-2$

$x^3 + \ldots + \frac{5}{2}(=0)$ OR $4x^3 + \ldots + 10(=0)$ ***(A1)(A1)***

Note: Award ***(A1)*** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

product of roots = $\left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right)$ OR $\left(\frac{(-1)^3 \times 10}{4}\right)$

$= -\frac{5}{2}$ ***A1***

continued…

Question 10 continued

OR

$4\left(x+\frac{1}{2}\right)=(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$

$x=-\frac{1}{2}$ ***(A1)***

or $4=x^2+\ldots+9 \Rightarrow x^2+\ldots+5=0$

product of roots of quadratic is 5 ***(A1)***

product is therefore $-\frac{1}{2}\times 5$

$=-\frac{5}{2}$ ***A1***

[4 marks]

Total [16 marks]

11. (a) attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division ***M1***

$3(-1)^3+5(-1)^2+(-1)-1=0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR
$$\begin{array}{r|l} & 3x^2+2x-1 \\ \hline x+1 & 3x^3+5x^2+x-1 \\ & 3x^3+3x^2 \\ \hline & \quad 2x^2+x \\ & \quad 2x^2+2x \\ \hline & \qquad -x-1 \\ & \qquad -x-1 \\ \hline & \qquad\quad 0 \end{array}$$
A1

[2 marks]

(b) attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division ***(M1)***

$P(x)=(x+1)(3x^2+2x-1)$ ***(A1)***

$=(x+1)(x+1)(3x-1)\left(=(x+1)^2(3x-1)\right)$ ***A1***

[3 marks]

(c) $\dfrac{1}{(x+1)(2x+1)}\equiv\dfrac{A}{x+1}+\dfrac{B}{2x+1}\Rightarrow 1\equiv A(2x+1)+B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\dfrac{1}{2}$ ***(M1)***

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\dfrac{1}{2}B$

$A=-1$ and $B=2$ ***A1A1***

Note: Award ***A1*** for each value.

$\dfrac{1}{(x+1)(2x+1)}=-\dfrac{1}{x+1}+\dfrac{2}{2x+1}$

[3 marks]

continued…

Question 11 continued

(d) $\dfrac{1}{(x+1)(x+1)(2x+1)}$

$=\dfrac{1}{(x+1)}\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)$ ***(A1)***

$=-\dfrac{1}{(x+1)^2}+\dfrac{2}{(2x+1)(x+1)}\left(=-\dfrac{1}{(x+1)^2}+2\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)\right)$ ***A1***

$=\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) attempt to integrate at least one term in $\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)\mathrm{d}x$

$=2\ln|2x+1|-2\ln|x+1|+\dfrac{1}{x+1}(+c)$ ***A1A1A1***

Note: Award ***A1*** for each correct term. Award a maximum of ***M1A1A0A1*** if modulus signs are omitted. Condone the absence of $+c$.

[4 marks]

(f) (i) **METHOD 1**

attempt to cancel factors and substitute $x=-1$ ***(M1)***

$\lim_{x\to-1} f(x)=\lim_{x\to-1}\left(\dfrac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right)=\lim_{x\to-1}\left(\dfrac{3x-1}{2x+1}\right)=\dfrac{3(-1)-1}{2(-1)+1}$

$=4$ ***A1***

continued…

Question 11 continued

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x=-1$ ***(M1)***

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right)=\lim_{x\to -1}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right)=\lim_{x\to -1}\left(\frac{18x+10}{12x+10}\right)=\frac{18(-1)+10}{12(-1)+10}$$

$=4$ ***A1***

(ii) **METHOD 1**

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{3x^3+\ldots}{2x^3+\ldots}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3+\text{terms which tend to 0}}{2+\text{terms which tend to 0}}\right)$$

$=\frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right)=\lim_{x\to\infty}\left(\frac{3x-1}{2x+1}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3-\frac{1}{x}}{2+\frac{1}{x}}\right)$$

$=\frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right)=\lim_{x\to\infty}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right)=\lim_{x\to\infty}\left(\frac{18x+10}{12x+10}\right)=\lim_{x\to\infty}\left(\frac{18}{12}\right)$$

$=\frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

Total [17 marks]

12. (a) attempt to expand the brackets or attempt to find modulus and argument of ϕ ***(M1)***

$(a+b\mathrm{i})^3 = a^3 + 3a^2b\mathrm{i} + 3a(b\mathrm{i})^2 + (b\mathrm{i})^3$ OR $\left(\sqrt{a^2+b^2}\right)^3 \operatorname{cis}\left(3\arctan\left(\frac{b}{a}\right)\right)$

(i) real part is $a^3 - 3ab^2$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

(ii) imaginary part is $3a^2b - b^3$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \sin\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

> **Note:** Award ***(M1)A1A0*** for $\left(a^3-3ab^2\right)+\left(3a^2b-b^3\right)\mathrm{i}$ OR
> $\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right) + \left(a^2+b^2\right)^{\frac{3}{2}} \mathrm{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$
>
> For (ii) condone $\left(3a^2b-b^3\right)\mathrm{i}$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \mathrm{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

[3 marks]

(b) attempt to substitute $a=1$ and $b=\sqrt{3}$ into their real or imaginary part found in (a) OR to expand the brackets OR to use polar form ***M1***

$(1-9)+\left(3\sqrt{3}-3\sqrt{3}\right)\mathrm{i}$ OR $\left(2\sqrt{3}\mathrm{i}-2\right)\left(1+\sqrt{3}\mathrm{i}\right) = 2\sqrt{3}\mathrm{i}-2-6-2\sqrt{3}\mathrm{i}$ OR

$\left(2\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)^3 = 8\mathrm{e}^{\mathrm{i}\pi}$ OR $\left(2\operatorname{cis}\left(60^\circ\right)\right)^3 = 8\operatorname{cis}\left(180^\circ\right)$ ***A1***

$=-8$ ***AG***

[2 marks]

(c) $v=-2,\ w=1-\sqrt{3}\mathrm{i}$ ***A1A1***

> **Note:** Award ***A1A0*** for $v=1-\sqrt{3}\mathrm{i},\ w=-2$ or if the labels v and w are not clearly specified or missing. Candidates may be awarded full ***FT*** marks for subsequent parts.

[2 marks]

continued…

Question 12 continued

(d) **METHOD 1**

triangle UVW has height $h = 3$ and base $b = 2\sqrt{3}$ ***(A1)***

attempt to find area of triangle with their height and base ***(M1)***

area = $\frac{1}{2} \times 2\sqrt{3} \times 3$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 2

triangle UVW has sides of length $\left(\sqrt{3^2 + \left(\sqrt{3}\right)^2} =\right) \sqrt{12}$ ***(A1)***

attempt to find area of equilateral triangle with their side length ***(M1)***

area = $\frac{1}{2}\left(\sqrt{12}\right)^2 \sin\frac{\pi}{3}$ OR $\frac{1}{2}\sqrt{12}(3)$ OR $\left(\sqrt{12}\right)^2 \times \frac{\sqrt{3}}{4}$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 3

triangle UVO has sides of length $\left(\sqrt{1^2 + \left(\sqrt{3}\right)^2} =\right) 2$ ***(A1)***

attempt to find area of three isosceles triangles with their side length and angle $\frac{2\pi}{3}$ ***(M1)***

area $= 3\left(\frac{1}{2}(2)^2 \sin\frac{2\pi}{3}\right)$

$= 3\sqrt{3}$ (square units) ***A1***

[3 marks]

continued…

Question 12 continued

(e) attempt to express u, v or w in the form $r\mathrm{e}^{\mathrm{i}\theta}$ and multiply by $\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$ ***(M1)***

$\left(u' = 2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}$ ***A1***

$\left(v' = 2\mathrm{e}^{\pi\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = 2\mathrm{e}^{\frac{5\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}}$ ***A1***

$\left(w' = 2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}}$ ***A1***

Note: These ***A1*** marks should be awarded independently and in any order.

[4 marks]

(f) **EITHER**

attempt to find one of $(u')^3, (v')^3$ or $(w')^3$ ***(M1)***

$(u')^3 = \left(2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}\right)^3 = 8\mathrm{e}^{\frac{7\pi}{4}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $(v')^3 = \left(2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}}\right)^3 = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR

$(w')^3 = \left(2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}}\right)^3 = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ ***(A1)***

OR

attempt to find product of their three roots u', v' and w' ***(M1)***

$u' \times v' \times w' = (c + d\,\mathrm{i})$

$2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}} \times 2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}} \times 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}} \times 2\mathrm{e}^{\frac{5\pi}{4}\mathrm{i}} \times 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ (or equivalent) ***(A1)***

OR

attempt to find $\left(z\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\right)^3$ for any z such that $z^3 = -8$ OR to rotate -8 by $\frac{3\pi}{4}$ ***(M1)***

$\left(\left(z\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\right)^3 = z^3\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}} =\right) -8\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}$ OR $8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $8\mathrm{cis}(-45^\circ)$ ***(A1)***

continued…

Question 12 continued

THEN

$8e^{-\frac{\pi}{4}i} = 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{8}{\sqrt{2}}(1-i)$

$= 4\sqrt{2} - 4\sqrt{2}i$ ***A1***

Note: Accept $c = \frac{8}{\sqrt{2}}, d = -\frac{8}{\sqrt{2}}$.

[3 marks]

(g) **METHOD 1**

attempt to write the arguments of u, v, w, u', v' and w' over a common denominator OR to write the arguments in degrees ***(M1)***

$\frac{-9\pi}{12}, \frac{-4\pi}{12}, \frac{-\pi}{12}, \frac{4\pi}{12}, \frac{7\pi}{12}, \frac{12\pi}{12}$ OR $-135^\circ, -60^\circ, -15^\circ, 60^\circ, 105^\circ, 180^\circ$

THEN

arguments of u, v, w, u', v' and w' differ by $\frac{3\pi}{12}$ and $\frac{5\pi}{12}$ OR 45° and 75°

so arguments of polygon vertices differ by $\frac{\pi}{12}$ or 15° ***(A1)***

$n = 24$ ***A1***

METHOD 2

Let $z = r\,\text{cis}\,\theta \Rightarrow z^n = r^n \text{cis}(n\theta) = r^n \text{cis}(n\theta)$, where θ is the argument of u, v, w, u', v' and w'.

recognition to find $n\theta$ where $n = 6, 12, 18, \ldots$ and $\theta = -\frac{3\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \pi$ ***(M1)***

when $n = 6 \Rightarrow (n\theta =) -\frac{9\pi}{2}, -2\pi, -\frac{\pi}{2}, 2\pi, \frac{7\pi}{2}, 6\pi$

when $n = 12 \Rightarrow (n\theta =) -9\pi, -4\pi, -\pi, 4\pi, 7\pi, 12\pi$ (which is not a multiple of 2π) ***(A1)***

$n = 24$ ***A1***

[3 marks]

Total [20 marks]

2224 –7106M

Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 1

31 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

SECTION A

1. $\tan^{-1}1 = 45^\circ$ or equivalent ***(A1)***

attempt to equate $2x - 5^\circ$ to their reference angle ***(M1)***

Note: Do not accept $2x - 5^\circ = 1$.

$2x - 5^\circ = 45^\circ, \left(225^\circ\right)$

$x = 25^\circ, 115^\circ$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.

[4 marks]

2. recognising a quadratic in 3^x ***(M1)***

$3\times\left(3^x\right)^2+5\times3^x-2=0$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) ***(M1)***

$\left(3\times3^x-1\right)\left(3^x+2\right)=0$ OR $3^x=\dfrac{-5\pm\sqrt{25+24}}{6}$ (or equivalent) ***(A1)***

$3^x=\dfrac{1}{3}$ (or $3^x=-2$) ***(A1)***

$x=-1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x=-1$ and $x=\log_3(-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

3. (a) (i) $\left(\frac{9}{2}, \frac{3\sqrt{3}}{2}\right)$ (accept $x = \frac{9}{2}$ and $y = \frac{3\sqrt{3}}{2}$) ***A1***

(ii) **METHOD 1**

using $m = \frac{\text{change in } y}{\text{change in } x}$ with their midpoint OR gradient perpendicular to AC

OR $m = \tan 30^\circ$ ***(M1)***

$m = \frac{\sqrt{3}}{3}$ ***(A1)***

$y = \frac{\sqrt{3}}{3}x$ OR $y - \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{9}{2}\right)$ (must be written as an equation) ***A1***

METHOD 2

attempt to find the vector equation of the line using a direction and a point on the line ***(M1)***

direction vector is $\begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ or equivalent/parallel vectors ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\boldsymbol{r} = \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ (or equivalent) ***A1***

> **Note:** Vector equation must be in the form $\boldsymbol{r} =$ or $\begin{pmatrix} x \\ y \end{pmatrix} =$.
>
> Allow equivalent parametric forms such as $x = \frac{9}{2}t$, $y = \frac{3\sqrt{3}}{2}t$.

[4 marks]

continued…

Question 3 continued.

(b) substituting $x = 6$ into their equation ***(M1)***

so at B $y = 2\sqrt{3}$ ***(A1)***

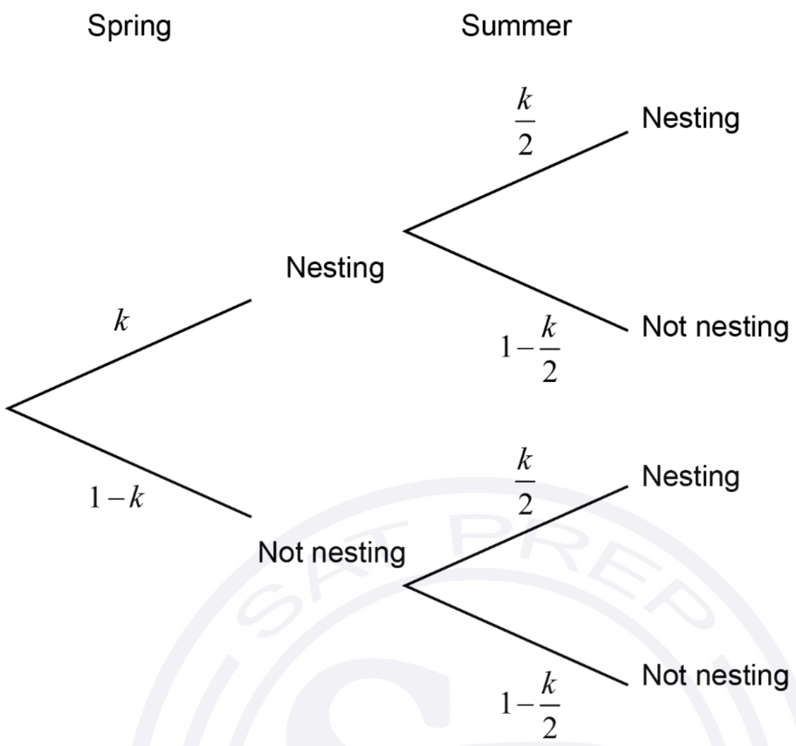
area of triangle $\text{OAB} = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$

area of quadrilateral $\text{OABC} = 12\sqrt{3}$ ***A1***

[3 marks]

Total [7 marks]

4. (a)



$1-k$ for Spring ***A1***

$1-\frac{k}{2}$ for both Summers ***A1***

[2 marks]

continued…

Question 4 continued.

(b) (i) multiplying the two correct branches **(A1)**

$(1-k)\left(1-\frac{k}{2}\right)$

attempt to expand and equate to $\frac{5}{9}$ **(M1)**

$1-k-\frac{k}{2}+\frac{k^2}{2}=\frac{5}{9}$

$18-18k-9k+9k^2=10$ OR $\frac{k^2}{2}-\frac{3k}{2}+\frac{4}{9}=0$ OR $\frac{9k^2}{2}-\frac{27k}{2}+4=0$ **A1**

$9k^2-27k+8=0$ **AG**

(ii) ($k=\frac{1}{3}$ is the only valid solution as) $\frac{8}{3}>1$ **R1**

Note: Accept any valid reasoning indicating that any probability cannot be greater than 1 and/or probability cannot be less than 0.

[4 marks]

Total [6 marks]

5. (a) $y = \frac{2}{3}$ (must be written as equation with $y =$) ***A1***

[1 mark]

(b) (i) 2 ***A1***

(ii) **EITHER**

$$\frac{2(x+3)}{3(x+2)} = mx + 1$$

attempt to expand to obtain a quadratic equation ***(M1)***

$2x + 6 = 3mx^2 + 6mx + 3x + 6$

$3mx^2 + (6m+1)x = 0$ OR $3mx^2 + 6mx + x = 0$ ***A1***

recognition that discriminant $\Delta = 0$ for one solution ***(M1)***

$(6m+1)^2 = 0$

continued…

Question 5 continued.

OR

$$\frac{2(x+3)}{3(x+2)} = mx+1$$

attempt to expand to obtain a quadratic equation ***(M1)***

$$2x+6 = 3mx^2+6mx+3x+6$$

$3mx^2+(6m+1)x=0$ OR $3mx^2+6mx+x=0$ ***A1***

attempt to solve their quadratic for x and equating their solutions ***(M1)***

$$x(3mx+6m+1)=0$$

$x=0$ OR $x=-\frac{6m+1}{3m}(=0)$

$$-\frac{6m+1}{3m}=0$$

OR

attempt to find $f'(x)$ using the quotient rule ***(M1)***

$f'(x)=\frac{2}{3}\left(\frac{(x+2)-(x+3)}{(x+2)^2}\right)=\left(\frac{-2}{3(x+2)^2}\right)$ OR $\frac{2(3x+6)-3(2x+6)}{(3x+6)^2}$ or equivalent ***A1***

recognition that m is the derivative of $f(x)$ at $x=0$ ***(M1)***

THEN

$\Rightarrow m=-\frac{1}{6}$ ***A1***

continued…

Question 5 continued.

(iii)

Note: In this part, FT may be awarded only for values of m between -1 and 0.

$-\frac{1}{6} < m < 0$ ***A2***

Note: Award ***A1*** for only $m > -\frac{1}{6}$. Award ***A1*** for only $m < 0$.

[7 marks]

Total [8 marks]

6. (a) $2.5(\%)$ ***A1***

[1 mark]

(b) $\text{P(weight of cooking apples} > 140) = 0.5\,(50\%)$ (seen anywhere) ***(A1)***

recognition of conditional probability in context ***(M1)***

$$\text{P}\left(\text{eating apple}|\text{weight} > 140\right) = \frac{\text{P(eating apple and weight of eating apple} > 140)}{\text{P(weight of apple} > 140)}$$

OR

$$\frac{\text{P(eating apple} \cap \text{weight of eating apple} > 140)}{\text{P(eating apple} \cap \text{weight of eating apple} > 140) + \text{P(cooking apple} \cap \text{weight of cooking apple} > 140)}$$

OR

$$\frac{\text{P}\left(\text{eating apple}\right)\text{P(weight of eating apple} > 140)}{\text{P}\left(\text{eating apple}\right)\text{P}\left(\text{weight of eating apple} > 140\right) + \text{P}\left(\text{cooking apple}\right)P(\text{weight of cooking apple} > 140)}$$

$$= \frac{0.8 \times 0.025}{0.8 \times 0.025 + 0.2 \times 0.5}\left(= \frac{80 \times 2.5}{80 \times 2.5 + 20 \times 50}\right)$$ ***(A1)***

$$= \frac{200}{1200}\left(= \frac{1}{6}\right)$$ ***A1***

Note: Accept any equivalent exact answer written as a fraction.

[4 marks]

Total [5 marks]

7. (a) $\alpha+\beta+\gamma=\frac{7}{2}$ ***A1***

[1 mark]

(b) $p-3\mathrm{i}$ is also a root (seen anywhere) ***A1***

recognition of 5 roots and attempt to sum these roots ***(M1)***

$p+3\mathrm{i}+p-3\mathrm{i}+\frac{7}{2}$

$p+3\mathrm{i}+p-3\mathrm{i}+\frac{7}{2}=\frac{11}{2}$ ***A1***

$p=1$ ***AG***

[3 marks]

(c) (i) attempt to find product of 5 roots and equate to ± 10 ***(M1)***

$(1+3\mathrm{i})(1-3\mathrm{i})\frac{1}{2}\alpha\beta=10$

$\alpha\beta=2$ ***A1***

(ii) $\alpha=1$ and $\beta=2$ ***A1***

[3 marks]

Total [7 marks]

8.

Note: To award full marks limit notation $\lim\limits_{x \to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1.***

$$\lim_{x \to 0} \frac{4\sec^4 x \tan x + 2\sin x \cos x}{4x^3 - 2x}$$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$$= \lim_{x \to 0} \frac{16\sec^4 x \tan^2 x + 4\sec^6 x - 2\sin^2 x + 2\cos^2 x}{12x^2 - 2}$$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x \to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$= -3$ ***A1***

[6 marks]

9. (a) $^{n}C_{3}$ ***A1***

[1 mark]

(b) **EITHER**

finding the number of ways to assign the students with the two students apart

number of ways to assign two students $^{2}C_{1}$ (seen anywhere) ***(A1)***

number of ways to assign others $^{n-2}C_{2}$ to have one group of 3 (seen anywhere) ***(A1)***

number of ways = $^{2}C_{1}\times{}^{n-2}C_{2}$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^{n}C_{3}={}^{2}C_{1}\times{}^{n-2}C_{2}$ OR ${}^{n}C_{3}=2\times{}^{2}C_{1}\times{}^{n-2}C_{2}$

valid attempt to eliminate all factorials from their equation ***(M1)***

$\frac{n(n-1)(n-2)}{3\times 2}=2\times 2\times\frac{(n-2)(n-3)}{2}$ or equivalent with no factorials

$n(n-1)=12(n-3)$

continued…

Question 9 continued.

OR

finding the number of ways to assign the students with the two students together

number of ways to assign two students and one other to the first group ${}^{n-2}C_1$
(seen anywhere) ***(A1)***

number of ways to assign three other students the first group ${}^{n-2}C_3$ (seen anywhere) ***(A1)***

number of ways $= {}^{n-2}C_1 + {}^{n-2}C_3$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}\,{}^{n}C_3 = {}^{n-2}C_1 + {}^{n-2}C_3$ OR ${}^{n}C_3 = 2\left({}^{n-2}C_1 + {}^{n-2}C_3\right)$

valid attempt to eliminate all factorials from their equation ***(M1)***

$$\frac{n(n-1)(n-2)}{3\times 2} = 2\times(n-2) + 2\times\frac{(n-2)(n-3)(n-4)}{3\times 2}$$

$$n(n-1) = 12 + 2(n-3)(n-4)$$

THEN

$n^2 - 13n + 36 = 0$ ***A1***

$(n-9)(n-4) = 0$

$n = 9$ ***A1***

Note: Do not award the final ***A1*** if additional values of n are given.

[6 marks]

Total [7 marks]

SECTION B

10. (a) attempt to find a difference *(M1)*

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation *A1*

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ *AG*

[2 marks]

(b) attempt to find a ratio *(M1)*

$r = \frac{s}{a}, r^2 = \frac{t}{a}, \ r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation *A1*

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ *AG*

[2 marks]

continued…

Question 10 continued

(c) **EITHER**

$2p-1=s^2$ (or equivalent) ***A1***

$\left(s^2>0\right) \Rightarrow\ 2p-1>0$ OR $s=\sqrt{2p-1}\Rightarrow 2p-1>0$ OR $p=\dfrac{s^2+1}{2}$ (and $s^2>0$) ***R1***

OR

$2p-1=a$ and $s^2=a$ ***A1***

($s^2>0$, so) $a>0\ \Rightarrow\ 2p-1>0$ OR $p=\dfrac{a+1}{2}$ and $a>0$ ***R1***

$\Rightarrow p>\dfrac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

continued…

Question 10 continued

(d) (i) $9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

(ii) $9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[4 marks]

(e) (i) attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR $d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2\ln 3$ OR $\ln 1 = 0$ OR $\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

continued…

Question 10 continued.

(ii) **METHOD 1**

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(2(9+2\ln 3)+9\left(-4-\ln 3\right)\right)$ (or equivalent) ***A1***

$=5\left(-18-5\ln 3\right)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

METHOD 2

$u_{10} = 9+\ln 9+9\left(-4-\ln 3\right)\left(=-27+\ln 9-9\ln 3\right)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(9+\ln 9-27+\ln 9-9\ln 3\right)$ OR

$\frac{10}{2}\left(2(9+2\ln 3)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(9+\ln 9-27-7\ln 3\right)$ (or equivalent) ***A1***

$=5\left(-18-5\ln 3\right)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

[6 marks]

Total [16 marks]

11.

(a) $2\times 2+6\times\frac{1}{2}-2\times 1=5$ ***A1***

[1 mark]

(b) $2\left(k^2-6\right)+6\left(2k+3\right)+12\left(=0\right)$ ***(A1)***

equating their scalar product of the direction normals to zero ***(M1)***

$2\left(k^2-6\right)+6\left(2k+3\right)+12=0$

$k^2-6+6k+9+6=0$ OR $\left(k+3\right)^2=0$

$k=-3$ ***A1***

attempt to substitute k, p and coordinates of A into Π_2 ***(M1)***

$q=3\times 2-3\times\frac{1}{2}-6\times 1$

$q=-\frac{3}{2}$ ***A1***

[5 marks]

continued…

Question 11 continued.

(c) attempt to equate a pair of ratios or equate vector product to zero vector ***(M1)***

$$\begin{pmatrix}2\\6\\-2\end{pmatrix}=\mu\begin{pmatrix}3\\9\\p\end{pmatrix}\Rightarrow\mu=\frac{2}{3}\text{ OR }\frac{2}{-2}=\frac{3}{p}\text{ OR }\frac{6}{-2}=\frac{9}{p}\text{ OR }6p+18=0\text{ OR }-6-2p=0$$

$p=-3$ ***A1***

[2 marks]

(d) (i) attempt to find the vector equation of the line through A perpendicular to Π_1 ***(M1)***

$$(\boldsymbol{r}=)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\\frac{1}{2}\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\6\\-2\end{pmatrix}$$

attempt to substitute their vector equation into Π_2 ***(M1)***

$$3(2+2\lambda)+9\left(\frac{1}{2}+6\lambda\right)-3(1-2\lambda)\left(=-\frac{51}{2}\right)$$

$6+6\lambda+\frac{9}{2}+54\lambda-3+6\lambda=-\frac{51}{2}$ (or equivalent) ***(A1)***

$66\lambda=-\frac{51}{2}-6-\frac{9}{2}+3$ (or equivalent)

$\lambda=-\frac{1}{2}$ ***A1***

$\left(1,-\frac{5}{2},2\right)$ ***A1***

continued…

Question 11 continued.

(ii) distance AB or $\left|\lambda\begin{pmatrix}2\\6\\-2\end{pmatrix}\right|$ ***(M1)***

$\sqrt{(2-1)^2+\left(\frac{1}{2}+\frac{5}{2}\right)^2+(1-2)^2}$ OR $\sqrt{1+9+1}$ ***A1***

$=\sqrt{11}$ ***AG***

[7 marks]

(e) Valid method to find a point C on Π_3 using $\overrightarrow{\text{AC}}=\overrightarrow{\text{BA}}$ or $\overrightarrow{\text{BC}}=2\overrightarrow{\text{BA}}$ or A as the midpoint of BC. ***(M1)***

$\lambda=\frac{1}{2}$, $\overrightarrow{\text{OC}}=\begin{pmatrix}2\\\frac{1}{2}\\1\end{pmatrix}+\begin{pmatrix}1\\3\\-1\end{pmatrix}, \overrightarrow{\text{OC}}=\begin{pmatrix}1\\-\frac{5}{2}\\2\end{pmatrix}+\begin{pmatrix}2\\6\\-2\end{pmatrix}, \overrightarrow{\text{OA}}=\frac{1}{2}\left(\overrightarrow{\text{OB}}+\overrightarrow{\text{OC}}\right)$

point on Π_3 is $\left(3,\frac{7}{2},0\right)$ ***A1***

attempt to substitute their $\left(3,\frac{7}{2},0\right)$ into $\Pi_3 : x+3y-z=d$ (or equivalent) ***(M1)***

$1\times 3+3\times\frac{7}{2}-1\times 0\left(=\frac{27}{2}\right)$

$\Pi_3 : x+3y-z=\frac{27}{2}$ $(2x+6y-2z=27)$ (or equivalent) ***A1***

[4 marks]

Total [19 marks]

12. (a) $n=1: LHS = f^{(1)}(x) = -\frac{1}{2}\times -a(1-ax)^{-\frac{3}{2}}\left(=\frac{a}{2}(1-ax)^{-\frac{3}{2}}\right)$ ***A1***

$RHS = \frac{a(1)!(1-ax)^{-\frac{3}{2}}}{2^1(0)!}$ therefore true for $n=1$ ***R1***

assume (that the result is) true for $n=k$ ***M1***

> **Note:** Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth must be clear.

$$f^{(k)}(x) = \frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}}}{2^{2k-1}(k-1)!}$$

attempt to differentiate the right-hand side with respect to x: ***M1***

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$$

$$= \frac{-(2k+1)\times -a}{2}\times\frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}-1}}{2^{2k-1}(k-1)!}$$ (or equivalent) ***A1***

attempt to multiply top and bottom by $2k$ ***M1***

$$= \frac{(2k+1)}{2}\times\frac{a^{k+1}(2k-1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k-1}(k-1)!}\times\frac{2k}{2k}$$

$$= \frac{a^{k+1}(2k+1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k+1}(k)!}$$ ***A1***

hence if the result is true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$ it is true for all $n\in\mathbb{Z}^+$ ***R1***

> **Note:** To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

continued…

Question 12 continued.

(b) $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \ldots$

$f''(x) = \frac{a^2(3)!(1-ax)^{-\frac{5}{2}}}{2^3(1)!}$ OR $\frac{3}{4}a^2(1-ax)^{-\frac{5}{2}}$ OR $f''(0) = \frac{a^2(3)!}{2^3}$ ***A1***

$f(0) = 1$, $f'(0) = \frac{a}{2}$, $f''(0) = \frac{6a^2}{8}$ ***A1***

$f(x) = 1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + \ldots.$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use $a=2$ or $a=4$ in the expansion **M1**

$$(1-2x)^{-\frac{1}{2}}=1+\frac{2x}{2}+\frac{3\times 4x^2}{8}\left(=1+x+\frac{3}{2}x^2+\ldots\right)$$

$$(1-4x)^{-\frac{1}{2}}=1+\frac{4x}{2}+\frac{3\times 16x^2}{8}\left(=1+2x+6x^2+\ldots\right)$$ **A1**

Note: Award ***A1*** for at least one correct.

attempt to multiply their two expansions together **M1**

$$\left(1+x+\frac{3}{2}x^2+\ldots\right)\left(1+2x+6x^2+\ldots\right)=1+2x+6x^2+x+2x^2+\frac{3}{2}x^2+\ldots$$

$=1+3x+\frac{19}{2}x^2+\ldots$ OR $\frac{2+4x+12x^2+2x+4x^2+3x^2+\ldots}{2}$ **A1**

$$(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}\approx\frac{2+6x+19x^2}{2}$$ **AG**

[4 marks]

(d) $|x|<\frac{1}{4}$ **A1**

[1 mark]

continued…

Question 12 continued.

(e)

$\left(\frac{2+6x+19x^2}{2}\right)=\frac{2+0.6+0.19}{2}\left(=\frac{279}{200}\right)$ (or equivalent) ***A1***

attempt to substitute $x=\frac{1}{10}$ into $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$ ***(M1)***

$\left((1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}=\right)\frac{10}{\sqrt{48}}$ (or equivalent) ***A1***

$\frac{10}{4\sqrt{3}}\approx\frac{279}{200}$ (or equivalent in terms of $\sqrt{3}$) ***A1***

$\frac{1}{\sqrt{3}}\approx\frac{279}{500}$

$\sqrt{3}\approx\frac{500}{279}$ ***A1***

[5 marks]

Total [20 marks]

8823 – 7111M

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

2. (a) $(\mathrm{P}(A \cup B) =) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = (A \cup B)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$(= 1 - 0.8)$

$= 0.2$ ***A1***

> **Note:** For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}(A' \cap B')$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3, \; q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1, \; d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =) 65 - 40$ OR $(u_5 =) 1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

4. **METHOD 1**

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2 - 1^2} =\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C} = 1 - \left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C} = \frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $= \frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$= \frac{1}{2} \times 10 \times \sqrt{6} \times \frac{\sqrt{24}}{5}$ ***(A1)***

$= 12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \text{B}\hat{\text{A}}\text{C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2 x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\dfrac{n(n-1)}{2} = 28$ OR $\dfrac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1***)

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \dfrac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \dfrac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

> **Note:** The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=(5^2)5^{2k}-(2^3)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8(5^{2k}-2^{3k})$ OR $25(5^{2k}-2^{3k})+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17(5^{2k}+8s)$ OR $17(25s+2^{3k})$ which is divisible by 17 ***A1***

OR

$(5^{2k}-2^{3k})\times5^2=5^{2k+2}-25\times2^{3k}=17s\times25$ ***M1***

$=5^{2k+2}-8\times2^{3k}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times2^{3k}=17s\times25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times25+17\times2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17(25s+2^{3k})$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

> **Note:** Only award ***R1*** if 4 of the previous 6 marks have been awarded
>
> **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

7. **METHOD 1**

attempt to substitute solution into given equation ***(M1)***

$(5+qi)^2 + i(5+qi) = -p + 25i$

$25 - q^2 + 10qi - q + 5i + p - 25i = 0$ OR $25 - q^2 + 10qi - q + 5i = -p + 25i$ ***A1***

$25 - q^2 + p - q + (10q - 20)i = 0$

attempt to equate real or imaginary parts: ***(M1)***

$10q - 20 = 0$ OR $25 - q^2 + p - q = 0$

$q = 2$, $p = -19$ ***A1A1***

METHOD 2

$z^2 + iz + p - 25i = 0$

sum of roots $= -i$, product of roots $= p - 25i$ ***M1***

one root is $(5+qi)$ so other root is $(-5-qi-i)$ ***A1***

product $(5+qi)(-5-qi-i) = -25 - 5qi - 5i - 5qi + q^2 + q = p - 25i$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25 = -10q - 5$ Re: $p = -25 + q^2 + q$

$q = 2$, $p = -19$ ***A1A1***

[5 marks]

8. (a) **METHOD 1**

attempt to integrate by parts **(M1)**

$u = (\ln x)^2$, $dv = x dx$ **(M1)**

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ dx$$ **A1**

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ **(M1)**

$$\int x \ln x \ dx = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$ **A1**

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$$ **A1**

[6 marks]

METHOD 2 (knowing $\int \ln x \ dx = x \ln x - x$ **)**

attempt to integrate by parts **(M1)**

$u = x \ln x$, $dv = \ln x dx$ **(M1)**

$$\int x \ln x (\ln x) \ dx = x \ln x (x \ln x - x) - \int (\ln x + 1)(x \ln x - x) dx$$ **A1**

$$= x \ln x (x \ln x - x) - \int x(\ln x)^2 dx + \int x dx$$ **A1**

$$2I = x \ln x (x \ln x - x) + \frac{x^2}{2} + c$$ **M1**

$$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$$ **A1**

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x \ln x \, dx$**)**

$$\int x \ln x \, dx = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x$, $dv = x \ln x dx$ ***(M1)***

$$\int (x \ln x) \ln x \, dx = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) dx$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int\left(\frac{x(\ln x)}{2} - \frac{x}{4}\right) dx$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2 \ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x)=\frac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$=\frac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$=\frac{\sin^2(kx)}{x^2}\left(=f(x)\right)$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+\ldots\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+\ldots\right)$ ***A1***

$$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+\ldots\right)}{x^2}\right)$$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in } x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits **(M1)**

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{k\cos kx}{1}\right)=k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx)=kx(+\ldots)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{kx+\ldots}{x}\right)=\lim_{x\to 0}(k+\text{terms in } x)=k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=k\times k=k^2$ ***A1***

$k^2=16\Rightarrow k=4\,(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\dfrac{x^2}{d}$ OR $\ln\dfrac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\dfrac{x^2}{d} = 2x-9$ OR $\dfrac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x =) \dfrac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10} \ (= p)$ or $x = 10 + \sqrt{10} \ (= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)e^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, ...$

$x = 0, \frac{\pi}{2}, \pi, ...$ ***A1***

Coordinates are $(0, e), \left(\frac{\pi}{2}, \frac{1}{e}\right), (\pi, e)$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.*** For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)e^{\cos 2x} - (4\cos 2x)e^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4e < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4e < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{e} > 0$ so minimum ***R1***

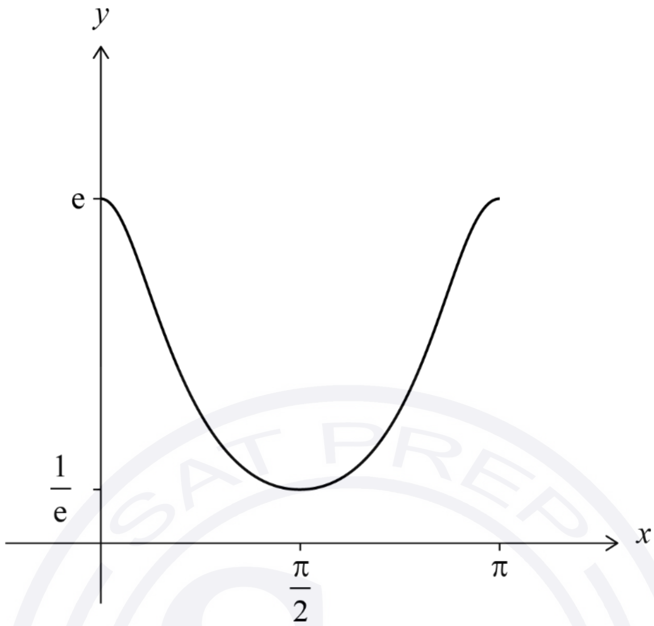
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + ...$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + ...;\ e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + ...$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + ...\right)\left(1 + \frac{2}{3}x^4 + ...\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x)\approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\dots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\dots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\,\mathrm{d}x\approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

> **Note:** Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta + {}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta + {}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta + {}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$$ ***A1***

$$= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + \mathrm{i}\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$$ ***A1A1***

> **Note:** Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ***A1***

substitute $\cos^2\theta = 1 - \sin^2\theta$ ***(M1)***

$\sin 5\theta = 5\left(1 - \sin^2\theta\right)^2\sin\theta - 10\sin^3\theta\left(1 - \sin^2\theta\right) + \sin^5\theta$ ***A1***

$\sin 5\theta = 5\left(1 - 2\sin^2\theta + \sin^4\theta\right)\sin\theta - 10\sin^3\theta\left(1 - \sin^2\theta\right) + \sin^5\theta$ ***A1***

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***AG***

> **Note:** Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \dfrac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \dfrac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\dfrac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \sqrt{\dfrac{5+\sqrt{5}}{8}}\sqrt{\dfrac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\dfrac{20}{64}}$ ***A1***

$= \dfrac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\dfrac{\pi}{5}, \sin\dfrac{2\pi}{5}, \sin\dfrac{3\pi}{5}, \sin\dfrac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\dfrac{5}{16}$ ***M1***

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$ ***A1***

recognition that $\sin\dfrac{\pi}{5} = \sin\dfrac{4\pi}{5}$ and $\sin\dfrac{2\pi}{5} = \sin\dfrac{3\pi}{5}$

$\sin^2\dfrac{\pi}{5}\sin^2\dfrac{3\pi}{5} = \dfrac{5}{16}$ ***A1***

$\sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \dfrac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

8823 – 7106M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

2. (a) $(\mathrm{P}(A \cup B) =) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = (A \cup B)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$(= 1 - 0.8)$

$= 0.2$ ***A1***

Note: For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}(A' \cap B')$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3$, $q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1$, $d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =)65 - 40$ OR $(u_5 =)1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

4. METHOD 1

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2 - 1^2} =\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C} = 1 - \left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C} = \frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $= \frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$= \frac{1}{2} \times 10 \times \sqrt{6} \times \frac{\sqrt{24}}{5}$ ***(A1)***

$= 12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \text{B}\hat{\text{A}}\text{C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5} \right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5} \right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2}\text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2 x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1***)

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=\left(5^2\right)5^{2k}-\left(2^3\right)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8\left(5^{2k}-2^{3k}\right)$ OR $25\left(5^{2k}-2^{3k}\right)+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17\left(5^{2k}+8s\right)$ OR $17\left(25s+2^{3k}\right)$ which is divisible by 17 ***A1***

OR

$\left(5^{2k}-2^{3k}\right)\times 5^2=5^{2k+2}-25\times 2^{3k}=17s\times 25$ ***M1***

$=5^{2k+2}-8\times 2^{3k}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times 2^{3k}=17s\times 25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times 25+17\times 2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17\left(25s+2^{3k}\right)$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if 4 of the previous 6 marks have been awarded **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

7. **METHOD 1**

attempt to substitute solution into given equation **(M1)**

$(5+q\mathrm{i})^2+\mathrm{i}(5+q\mathrm{i})=-p+25\mathrm{i}$

$25-q^2+10q\mathrm{i}-q+5\mathrm{i}+p-25\mathrm{i}=0$ OR $25-q^2+10q\mathrm{i}-q+5\mathrm{i}=-p+25\mathrm{i}$ **A1**

$25-q^2+p-q+(10q-20)\mathrm{i}=0$

attempt to equate real or imaginary parts: **(M1)**

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2$, $p=-19$ **A1A1**

METHOD 2

$z^2+\mathrm{i}z+p-25\mathrm{i}=0$

sum of roots $=-\mathrm{i}$, product of roots $=p-25\mathrm{i}$ **M1**

one root is $(5+q\mathrm{i})$ so other root is $(-5-q\mathrm{i}-\mathrm{i})$ **A1**

product $(5+q\mathrm{i})(-5-q\mathrm{i}-\mathrm{i})=-25-5q\mathrm{i}-5\mathrm{i}-5q\mathrm{i}+q^2+q=p-25\mathrm{i}$

equating real and imaginary parts for product of roots **(M1)**

Im: $-25=-10q-5$ Re: $p=-25+q^2+q$

$q=2$, $p=-19$ **A1A1**

[5 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \, \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \, \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \, \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \, \mathrm{d}x\right]$ ***A1***

$\int x(\ln x)^2 \, \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \, \mathrm{d}x\right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \, \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x\mathrm{d}x$ ***(M1)***

$\int x \ln x(\ln x) \, \mathrm{d}x = x \ln x(x \ln x - x) - \int (\ln x + 1)(x \ln x - x)\mathrm{d}x$ ***A1***

$= x \ln x(x \ln x - x) - \int x(\ln x)^2 \mathrm{d}x + \int x\mathrm{d}x$ ***A1***

$2I = x \ln x(x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\,\mathrm{d}x$**)**

$$\int x\ln x\,\mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x$, $\mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$$\int (x\ln x)\ln x\,\mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int\left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x)=\frac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$=\frac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$=\frac{\sin^2(kx)}{x^2}(=f(x))$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim\limits_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim\limits_{x\to 0}\left(f\left(x\right)\right)=\lim\limits_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim\limits_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$=\lim\limits_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim\limits_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim\limits_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+\ldots\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+\ldots\right)$ ***A1***

$\lim\limits_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim\limits_{x\to 0}\left(\frac{k^2x^2\left(+\ldots\right)}{x^2}\right)$ ***M1***

$=\lim\limits_{x\to 0}\left(k^2+\text{terms in }x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim\limits_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits (***M1)***

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{k\cos kx}{1}\right)=k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx)=kx(+...)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{kx+...}{x}\right)=\lim_{x\to 0}(k+\text{terms in } x)=k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=k\times k=k^2$ ***A1***

$k^2=16\Rightarrow k=4(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\frac{x^2}{d} = 2x-9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9, \infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x =) \dfrac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10} \, (= p)$ or $x = 10 + \sqrt{10} \, (= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)e^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, e), \left(\frac{\pi}{2}, \frac{1}{e}\right), (\pi, e)$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***   For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)e^{\cos 2x} - (4\cos 2x)e^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4e < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4e < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{e} > 0$ so minimum ***R1***

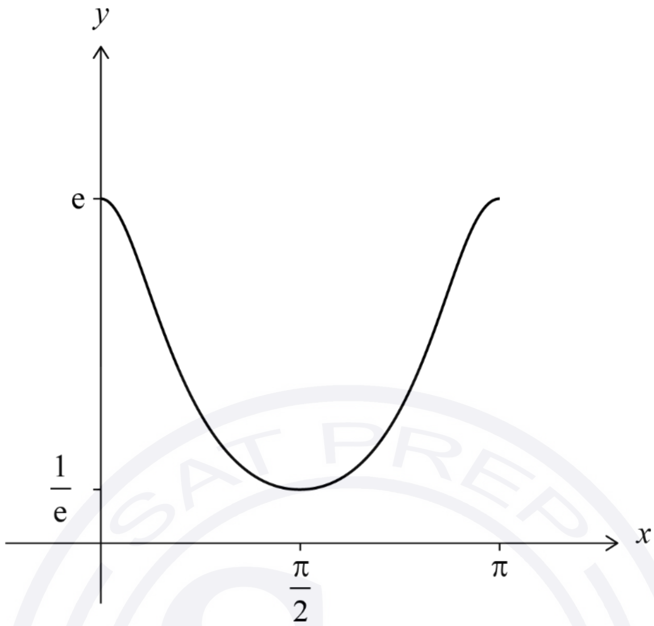
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$$ ***A1***

$$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$$

$$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$$ ***A1***

METHOD 2

$$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots; \quad e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$$

$$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$$ ***A1***

$$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x)\approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\,\mathrm{d}x\approx\mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note:	If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$ ***A1***

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right)$$ ***A1A1***

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos 5\theta+\mathrm{i}\sin 5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ ***A1***

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$\sin 5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$\sin 5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \dfrac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \dfrac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\dfrac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \sqrt{\dfrac{5+\sqrt{5}}{8}}\sqrt{\dfrac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\dfrac{20}{64}}$ ***A1***

$= \dfrac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\dfrac{\pi}{5}, \sin\dfrac{2\pi}{5}, \sin\dfrac{3\pi}{5}, \sin\dfrac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\dfrac{5}{16}$ ***M1***

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$ ***A1***

recognition that $\sin\dfrac{\pi}{5} = \sin\dfrac{4\pi}{5}$ and $\sin\dfrac{2\pi}{5} = \sin\dfrac{3\pi}{5}$

$\sin^2\dfrac{\pi}{5}\sin^2\dfrac{3\pi}{5} = \dfrac{5}{16}$ ***A1***

$\sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \dfrac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 1

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x}\times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x}\times e^{3x}-e^{4x}\times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempts to find perimeter ***(M1)***

$\text{arc} + 2 \times \text{radius}$ OR $10 + 4 + 4$

$= 18 \text{ (cm)}$ ***A1***

[2 marks]

(b) $10 = 4\theta$ ***(A1)***

$\theta = \frac{10}{4} \left(= \frac{5}{2}, 2.5\right)$ ***A1***

[2 marks]

(c) area $= \frac{1}{2}\left(\frac{10}{4}\right)\left(4^2\right) \ (= 1.25 \times 16)$ ***(A1)***

$= 20 \text{ (cm}^2\text{)}$ ***A1***

[2 marks]

Total [6 marks]

2. (a) (i) $x = 2$ ***A1***

(ii) $y = 1$ ***A1***

[2 marks]

(b) (i) $\left(0, \frac{3}{2}\right)$ ***A1***

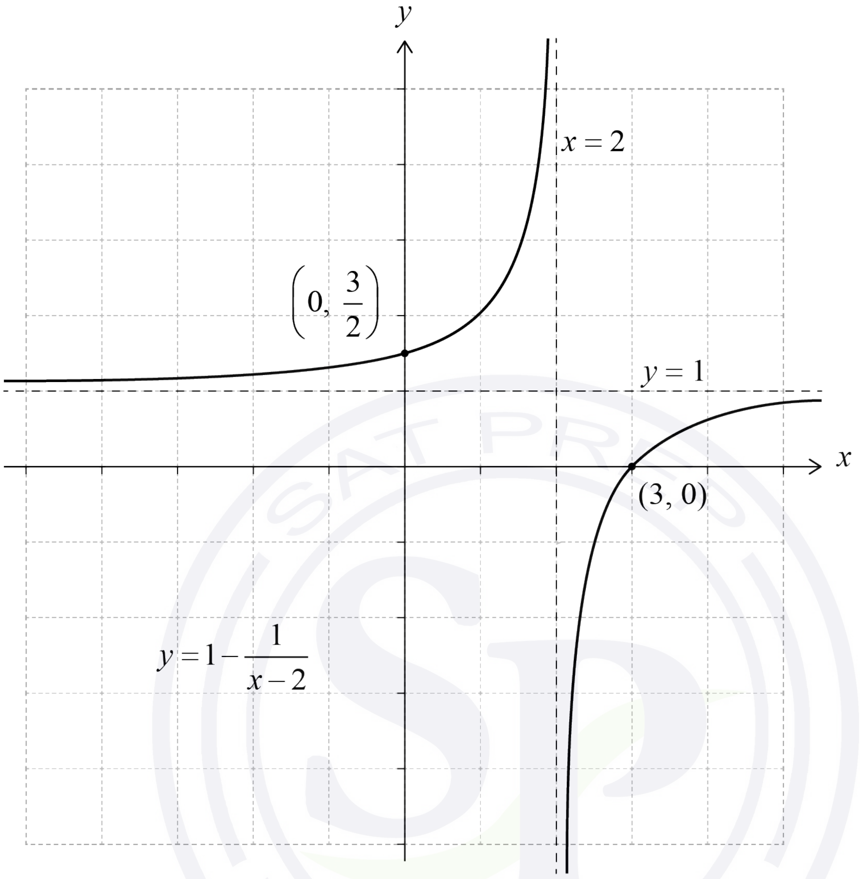
(ii) $(3, 0)$ ***A1***

[2 marks]

continued…

Question 2 continued

(c)



two correct branches with correct asymptotic behaviour and intercepts clearly shown ***A1***

[1 mark]

Total [5 marks]

3. substitutes into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ to form

$0.55 = 0.4 + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ (or equivalent) ***(A1)***

substitutes into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ to form $0.25 = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ (or equivalent) ***(A1)***

attempts to combine their two probability equations to form an equation in $\mathrm{P}(B)$ ***(M1)***

Note: The above two ***A*** marks are awarded independently.

correct equation in $\mathrm{P}(B)$ ***A1***

$0.55 = 0.4 + \mathrm{P}(B) - 0.25\mathrm{P}(B)$ OR $\dfrac{\mathrm{P}(B) - 0.15}{\mathrm{P}(B)} = 0.25$ OR $\mathrm{P}(B) - 0.15 = 0.25\mathrm{P}(B)$

(or equivalent)

$\mathrm{P}(B) = \dfrac{15}{75} \left(= \dfrac{1}{5} = 0.2 \right)$ ***A1***

Total [5 marks]

4. $A=\int_0^c \frac{x}{x^2+2}\,\mathrm{d}x$

EITHER

attempts to integrate by inspection or substitution using $u=x^2+2$ or $u=x^2$ ***(M1)***

Note: If candidate simply states $u=x^2+2$ or $u=x^2$, but does not attempt to integrate, do not award the ***(M1)*** .

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k\ln u$ or $k\ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln(u+2)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k\ln(x^2+2)$.

$\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln(c^2+2)-\ln 2\right)(=\ln 3)$ OR $\frac{1}{2}\ln(c^2+2)-\frac{1}{2}\ln 2(=\ln 3)$

continued…

Question 4 continued

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)(=\ln 3)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}\,(=\ln 3)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln\left(c^2+2\right)-\ln 2=\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}\,(=\ln 3)$ OR $\ln\frac{\sqrt{c^2+2}}{\sqrt{2}}\,(=\ln 3)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ OR $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c=\pm 4$ as a final answer.

Total [6 marks]

5. attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f(x) + 3$ OR $(ax+b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3 \left(= 4x^2 - 14x + 15\right)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR $2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

[7 marks]

6. (a) $\mathrm{E}(X) = 2a$ (by symmetry) ***A1***

[1 mark]

(b) **METHOD 1**

uses $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{x^2}{2a} \mathrm{d}x - (2a)^2$$

$$= \left[\frac{x^3}{6a}\right]_a^{3a} - (2a)^2$$ ***A1***

$$= \frac{13a^2}{3} - (2a)^2$$ ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

Note: Award as above if $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$ are calculated separately leading to $\mathrm{Var}(X) = \frac{a^2}{3}$. Award ***(M1)A0A0A0*** for $\mathrm{Var}(X) = \frac{13a^2}{3}$.

continued…

Question 6 continued

METHOD 2

uses $\mathrm{Var}(X) = \mathrm{E}(X - \mathrm{E}(X))^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{(x-2a)^2}{2a}\,\mathrm{d}x$$

$$= \left[\frac{(x-2a)^3}{6a}\right]_a^{3a}$$ ***A1***

$$= \frac{a^3 - (-a^3)}{6a}$$ or equivalent ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

[4 marks]

Total [5 marks]

7. let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\frac{r}{(r+1)!}=1-\frac{1}{(n+1)!}$ for all integers, $n\geq 1$

considering $\mathrm{P}(1)$:

$\mathrm{LHS}=\frac{1}{2}$ and $\mathrm{RHS}=\frac{1}{2}$ and so $\mathrm{P}(1)$ is true **_R1_**

assume $\mathrm{P}(k)$ is true *ie*, $\sum_{r=1}^{k}\frac{r}{(r+1)!}=1-\frac{1}{(k+1)!}$ **_M1_**

> **Note:** Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\frac{r}{(r+1)!}=\sum_{r=1}^{k}\frac{r}{(r+1)!}+\frac{k+1}{((k+1)+1)!}$$ ***(M1)***

$$=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$$ ***A1***

$$=1-\frac{(k+2)-(k+1)}{(k+2)!}$$ ***A1***

$$=1-\frac{1}{(k+2)!}\left(=1-\frac{1}{((k+1)+1)!}\right)$$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true (for all integers, $n\geq 1$) **_R1_**

> **Note:** To obtain the final ***R1***, any four of the previous marks must have been awarded.

[7 marks]

8. (a) $\cos k = \dfrac{\sin k}{\cos k}$ ***A1***

$\cos^2 k = \sin k$ ***AG***

[1 mark]

(b) $f'(k) = -\sin k$ and $g'(k) = \sec^2 k$ ***A1***

Note: Award ***A1*** for $f'(x) = -\sin x$ and $g'(x) = \sec^2 x$.

EITHER

$f'(k)g'(k) = -\dfrac{\sin k}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow f'(k)g'(k)\left(= -\dfrac{\sin k}{\sin k}\right) = -1$ ***R1***

OR

$g'(k) = \dfrac{1}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow g'(k) = \dfrac{1}{\sin k} = -\dfrac{1}{f'(k)}$ ***R1***

Note: Accept showing that $f'(k) = -\dfrac{1}{g'(k)}$.

Note: Allow'backwards methods' such as starting with $f'(k) = -\dfrac{1}{g'(k)}$ leading to $\cos^2 k = \sin k$

THEN

$\Rightarrow$ the two tangents intersect at right angles at P ***AG***

Note: To obtain the final ***R1***, all of the previous marks must have been awarded.

[3 marks]

continued…

Question 8 continued

(c) $1-\sin^2 k = \sin k$ (from part (a)) ***A1***

$\sin^2 k + \sin k - 1 = 0$

attempts to solve for $\sin k$ ***(M1)***

$\sin k = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$

(for $0 < k < \dfrac{\pi}{2}$, $\sin k > 0$) $\Rightarrow \sin k = \dfrac{-1+\sqrt{5}}{2}$ ***A1***

$(a = -1, b = 5, c = 2)$

Note: Award ***A0*** if more than one solution is given

[3 marks]

Total [7 marks]

9. (a) $\overrightarrow{OM} = \boldsymbol{a} + k\boldsymbol{c}$ ***A1***

$\overrightarrow{MC} = (1-k)\boldsymbol{c} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) attempts to expand their dot product $\overrightarrow{OM} \bullet \overrightarrow{MC} = (\boldsymbol{a} + k\boldsymbol{c}) \bullet ((1-k)\boldsymbol{c} - \boldsymbol{a})$ ***M1***

$= (1-2k)(\boldsymbol{a} \bullet \boldsymbol{c}) - |\boldsymbol{a}|^2 + k(1-k)|\boldsymbol{c}|^2$ (or equivalent)

uses $|\boldsymbol{c}| = 2|\boldsymbol{a}|$ and $\boldsymbol{a} \bullet \boldsymbol{c} = 2|\boldsymbol{a}|^2 \cos\theta$ ***M1***

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - |\boldsymbol{a}|^2 + 4k(1-k)|\boldsymbol{a}|^2$

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - (1-2k)^2 |\boldsymbol{a}|^2$ ***A1***

$|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ ***AG***

[3 marks]

(c) attempts to solve $|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ for k ***(M1)***

$k = \frac{1}{2}$ or $k = \frac{1}{2} - \cos\theta$ $\left(|\boldsymbol{a}|^2 > 0\right)$

> **Note:** Award ***(M1)*** for their '$k =$' or their '$\cos\theta =$'. For example, $\cos\theta = \frac{1-2k}{2}$ or equivalent.

as $0 \le k \le 1$, $0 \le \frac{1}{2} - \cos\theta \le 1$

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$ ***A1***

$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\theta \ne \frac{\pi}{2}$ ***A1A1***

($\theta = \frac{\pi}{2}$ corresponds to only one possible position for M when $k = \frac{1}{2}$)

[4 marks]

Total [9 marks]

Section B

10. (a) $y^2 = 9 - x^2$ OR $y = \pm\sqrt{9 - x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9 - x^2}$ ***AG***

[1 mark]

(b) $b = 2y$ $\left(= 2\sqrt{9 - x^2}\right)$ or $h = x + 3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \dfrac{1}{2}bh$ ***(M1)***

$A = \sqrt{9 - x^2}\,(x + 3)$ (or equivalent) $\left(= \dfrac{2(x+3)\sqrt{9 - x^2}}{2} = x\sqrt{9 - x^2} + 3\sqrt{9 - x^2}\right)$ ***A1***

[3 marks]

(c) **METHOD 1**

attempts to use the product rule to find $\dfrac{dA}{dx}$ ***(M1)***

attempts to use the chain rule to find $\dfrac{d}{dx}\sqrt{9 - x^2}$ ***(M1)***

$\left(\dfrac{dA}{dx} =\right) \sqrt{9 - x^2} + (3 + x)\left(\dfrac{1}{2}\right)\left(9 - x^2\right)^{-\frac{1}{2}}(-2x)$ $\left(= \sqrt{9 - x^2} - \dfrac{x^2 + 3x}{\sqrt{9 - x^2}}\right)$ ***A1***

$\left(\dfrac{dA}{dx} =\right) \dfrac{9 - x^2}{\sqrt{9 - x^2}} - \dfrac{x^2 + 3x}{\sqrt{9 - x^2}}$ $\left(= \dfrac{9 - x^2 - \left(x^2 + 3x\right)}{\sqrt{9 - x^2}}\right)$ ***A1***

$\dfrac{dA}{dx} = \dfrac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$ ***AG***

continued…

Question 10 continued

METHOD 2

$$\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$$

attempts to find $\frac{dA}{dy}$ where $A = y(x+3)$ and $\frac{dy}{dx}$ where $y^2 = 9 - x^2$ ***(M1)***

$\frac{dA}{dy} = y\frac{dx}{dy} + x + 3$ and $\frac{dy}{dx} = -\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{dA}{dy}$ and their $\frac{dy}{dx}$ into $\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$ ***(M1)***

$\frac{dA}{dx} = \left(y\left(-\frac{y}{x}\right) + x + 3 \right)\left(-\frac{x}{y}\right)$ (or equivalent)

$= \frac{9 - x^2 - x^2 - 3x}{\sqrt{9 - x^2}}$ (or equivalent) ***A1***

$\frac{dA}{dx} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$ ***AG***

[4 marks]

continued…

Question 10 continued

(d) $\frac{\mathrm{d}A}{\mathrm{d}x}=0\ \left(\frac{9-3x-2x^2}{\sqrt{9-x^2}}=0\right)$ ***(M1)***

attempts to solve $9-3x-2x^2=0$ (or equivalent) ***(M1)***

$-(2x-3)(x+3)(=0)$ OR $x=\frac{3\pm\sqrt{(-3)^2-4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x=\frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x=-3$ is also given.

substitutes their value of x into either $y=\sqrt{9-x^2}$ or $y=-\sqrt{9-x^2}$ ***(M1)***

Note: Do not award the above ***(M1)*** if $x\le 0$.

$y=-\sqrt{9-\left(\frac{3}{2}\right)^2}$

$=-\frac{\sqrt{27}}{2}\left(=-\frac{3\sqrt{3}}{2},=-\sqrt{\frac{27}{4}},=-\sqrt{6.75}\right)$ ***A1***

[6 marks]

Total [14 marks]

11. (a) **METHOD 1**

$|u| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \left(= \sqrt{1+3}\right)$ ***A1***

$= 2$ ***AG***

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

> **Note:** Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ and so $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ ***AG***

[3 marks]

METHOD 2

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

> **Note:** Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ ***AG***

$r\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}\mathrm{i}$

$r = \frac{-1}{\cos\frac{2\pi}{3}} = \frac{-1}{-\frac{1}{2}}$ OR $r = \frac{\sqrt{3}}{\sin\frac{2\pi}{3}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}}$ ***A1***

$= 2$ and so $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ ***AG***

[3 marks]

continued…

Question 11 continued

(b) (i) $u^n \in \mathbb{R} \Rightarrow \frac{2n\pi}{3} = k\pi \ \left(k \in \mathbb{Z}\right)$ ***(M1)(A1)***

Note: Award ***M1*** for noting that $\sin\frac{2n\pi}{3} = 0$ from $u^n = 2^n\left(\cos\frac{2n\pi}{3} + \mathrm{i}\sin\frac{2n\pi}{3}\right)$.
Award ***(A1)*** for a multiple of 3 considered.

$n = 3$ ***A1***

(ii) substitutes their value (must be a multiple of 3) for n into u^n ***(M1)***

$u^3 = 2^3\cos 2\pi$

$= 8$ ***A1***

[5 marks]

continued…

Question 11 continued

(c) (i) $-1-\sqrt{3}\mathrm{i}$ is a root (by the conjugate root theorem) ***A1***

Note: Accept $2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

let $z=c$ be the real root

EITHER

uses sum of roots (equated to ± 5) ***(M1)***

$\left(\left(-1+\sqrt{3}\mathrm{i}\right)+\left(-1-\sqrt{3}\mathrm{i}\right)+c\right)=-5$ ***(A1)***

$-2+c=-5$ ***(A1)***

OR

uses product of roots (equated to ± 12) ***(M1)***

$\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)c=-12$ ***(A1)***

$4c=-12$ ***(A1)***

OR

$\left(z-\left(-1+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(-1-\sqrt{3}\mathrm{i}\right)\right)=z^2+2z+4$ ***(A1)***

compares coefficients *eg* ***(M1)***

$(z-c)\left(z^2+2z+4\right)=z^3+5z^2+10z+12$

$-4c=12$ ***(A1)***

THEN

$c=-3$ (and so $z=-3$ is a root) ***A1***

continued…

Question 11 continued

(ii) **METHOD 1**

compares $z^3+5z^2+10z+12=0$ and $1+5w+10w^2+12w^3=0$

$z=\frac{1}{w}\Rightarrow w=\frac{1}{z}$ ***A2***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

METHOD 2

attempts to factorize into a product of a linear factor and a quadratic factor ***(M1)***

$1+5w+10w^2+12w^3=(3w+1)(4w^2+2w+1)$ ***A1***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

[9 marks]

(d) $(a+b\mathrm{i})^2=2(a-b\mathrm{i})$ ***A1***

attempts to expand and equate real and imaginary parts: ***M1***

$a^2-b^2+2ab\mathrm{i}=2a-2b\mathrm{i}$

$a^2-b^2=2a$ and $2ab=-2b$

attempts to find the value of a or b ***M1***

$2b(a+1)=0$

$b=0\Rightarrow a^2=2a\Rightarrow a=2$ (real root) ***A1***

$a=-1\Rightarrow 1-b^2=-2\Rightarrow b=\pm\sqrt{3}$ (complex roots $-1\pm\sqrt{3}\mathrm{i}$) ***A1***

[5 marks]

Total [22 marks]

12. (a) let $t=\sqrt{x}$ ***M1***

$t^2=x \Rightarrow 2t\ \mathrm{d}t=\mathrm{d}x$ (or equivalent) ***A1***

so $\int \cos\sqrt{x}\ \mathrm{d}x = 2\int t\cos t\ \mathrm{d}t$ ***A1***

attempts integration by parts ***(M1)***

$u=2t$, $\mathrm{d}v=\cos t\ \mathrm{d}t$, $\mathrm{d}u=2\ \mathrm{d}t$, $v=\sin t$

$2\int t\cos t\ \mathrm{d}t = 2t\sin t - 2\int \sin t\ \mathrm{d}t$ ***(A1)***

$=2t\sin t + 2\cos t + C$ ***A1***

substitution of $t=\sqrt{x}$ $\Rightarrow \int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ ***AG***

[6 marks]

(b) $x_{n+1}=\dfrac{\left(2(n+1)-1\right)^2\pi^2}{4}\ \left(=\dfrac{(2n+1)^2\pi^2}{4}\right)$ ***A1***

[1 mark]

continued…

Question 12 continued

(c) area of R_n is $\left|\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x\right|$ ***(M1)***

Note: Modulus may be seen at a later stage.

$$=\left|\left[2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}\right|$$ ***A1***

Note: Condone $+C$ at this stage.

attempts to substitute their limits into their integrated expression ***(M1)***

$$=2\left|\frac{(2n+1)\pi}{2}\times\sin\frac{(2n+1)\pi}{2}+\cos\frac{(2n+1)\pi}{2}-\left(\frac{(2n-1)\pi}{2}\times\sin\frac{(2n-1)\pi}{2}+\cos\frac{(2n-1)\pi}{2}\right)\right|$$

A1

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}-\left((-1)^{n+1}\frac{(2n-1)\pi}{2}\right)\right|$$ (or equivalent) ***A1***

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}+(-1)^n\frac{(2n-1)\pi}{2}\right|$$ ***A1***

$$=2\left|(-1)^n\frac{4n\pi}{2}\right|$$

$$=4n\pi$$ ***A1***

Note: Award a maximum of ***(M1)A1M1A1A1A0A0*** for only attempting to calculate $\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x$, and not applying the modulus.

[7 marks]

continued…

Question 12 continued

(d) **EITHER**

attempts to find $(d =)R_{n+1} - R_n$ ***M1***

$(d =)4(n+1)\pi - 4n\pi$

$= 4\pi$ ***A1***

Note: Award ***M0*** for consideration of special cases for example R_3 and R_2.
Accept $d = k\pi$.

which is a constant (common difference is 4π) ***R1***

OR

an arithmetic sequence is of the form $u_n = dn + c$ $(u_n = dn + u_1 - d)$ ***M1***

attempts to compare $u_n = dn + c$ $(u_n = dn + u_1 - d)$ and $R_n = 4n\pi$ ***M1***

$d = 4\pi$ and $c = 0$ $(u_1 - d = 0)$ ***A1***

Note: Accept $d = k\pi$.

THEN

so the areas of the regions form an arithmetic sequence ***AG***

[3 marks]

Total [17 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 1

31 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) recognizing $f(x)=0$ ***(M1)***

$x=-1$ ***A1***

[2 marks]

(b) (i) $x=2$ (must be an equation with x) ***A1***

(ii) $y=\frac{7}{2}$ (must be an equation with y) ***A1***

[2 marks]

(c) **EITHER**

interchanging x and y ***(M1)***

$2xy-4x=7y+7$

correct working with y terms on the same side: $2xy-7y=4x+7$ ***(A1)***

OR

$2yx-4y=7x+7$

correct working with x terms on the same side: $2yx-7x=4y+7$ ***(A1)***

interchanging x and y OR making x the subject $x=\frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x)=\frac{4x+7}{2x-7}$ (or equivalent) $\left(x\neq\frac{7}{2}\right)$ ***A1***

[3 marks]

Total [7 marks]

2. (a) (i) summing frequencies of riders or finding complement *(M1)*

probability $= \frac{34}{40}$ ***A1***

(ii) attempt to find expected value *(M1)*

$$\frac{16}{40}+\left(2\times\frac{13}{40}\right)+\left(3\times\frac{2}{40}\right)+\left(4\times\frac{3}{40}\right)$$

$\frac{60}{40}(=1.5)$ ***A1***

[4 marks]

(b) evidence of **their** rides/visitor $\times 1000$ or $\div 10$ *(M1)*

1500 OR 0.15

150 (times) ***A1***

[2 marks]

Total [6 marks]

3. $1-2\sin^2 x=\sin x$ ***A1***

$2\sin^2 x+\sin x-1=0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x-1)(\sin x+1)$ OR $\frac{-1\pm\sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x=\frac{1}{2}$ OR $\sin x=-1$

any correct solution from $\sin x=-1$ ***A1***

any correct solution from $\sin x=\frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x=-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).
Award ***A0*** for $-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ if additional values also given.

Total [6 marks]

4. recognition of quadratic in e^x ***(M1)***

$\left(\mathrm{e}^x\right)^2 - 3\mathrm{e}^x + \ln k\left(=0\right)$ OR $A^2 - 3A + \ln k\left(=0\right)$

recognizing discriminant ≥ 0 (seen anywhere) ***(M1)***

$\left(-3\right)^2 - 4(1)(\ln k)$ OR $9 - 4\ln k$ ***(A1)***

$\ln k \le \dfrac{9}{4}$ ***(A1)***

$\mathrm{e}^{9/4}$ (seen anywhere) ***A1***

$0 < k \le \mathrm{e}^{9/4}$ ***A1***

[6 marks]

5. (a) recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

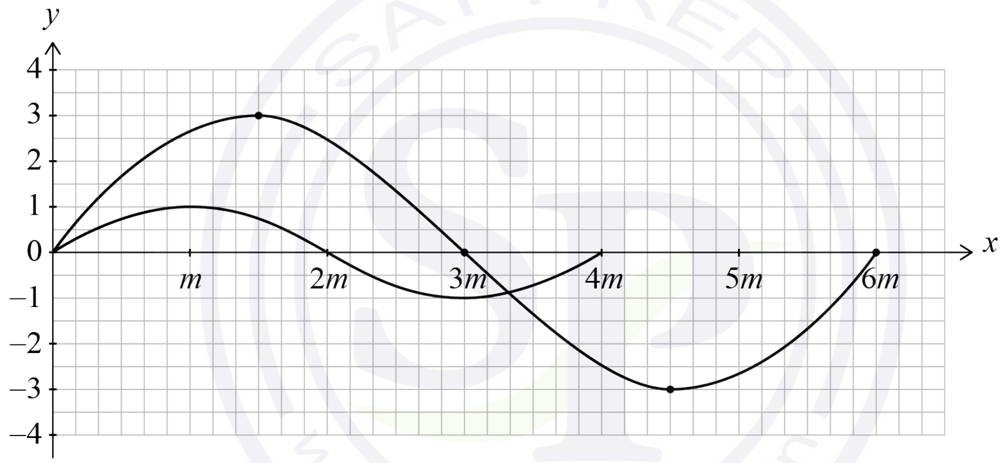
$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

(b) horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note: Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

Total [6 marks]

6. $A=\frac{1}{2}x^2\sin\frac{\pi}{3}$ OR $A=\frac{1}{2}x^2\sin 60^\circ$ OR triangle height $h=\sqrt{x^2-\left(\frac{x}{2}\right)^2}$ $\left(=\frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$=\frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A=\frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(=\frac{\sqrt{3}}{4}x^2\right)$ ***A1***

> **Note:** Award ***A1*** for $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$\frac{dA}{dt}=\frac{\sqrt{3}}{4}\times 2x\frac{dx}{dt}$ OR $\frac{dA}{dt}=\frac{1}{2}\times\sin\frac{\pi}{3}\times 2x\frac{dx}{dt}$ ***(A1)***

$=\frac{2\sqrt{3}}{4}\times 5\sqrt{3}\times 4$

$\frac{dA}{dt}=30\,(\text{cm}^2\text{s}^{-1})$ ***A1***

> **Note:** Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

7. **METHOD 1**

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

> **Note:** Award ***A1A1*** for $P(3\text{i})=0$ and $P(-3\text{i})=0$ seen in their working.
> Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\text{i})=0$ or $P(-3\text{i})=0$ ***(M1)***

$$4m-3m\text{i}+\frac{36}{m}(3\text{i})^2-(3\text{i})^3=0$$

$$4m-3m\text{i}-\frac{36}{m}(-9)+27\text{i}=0$$

attempt to equate the real or imaginary parts ***(M1)***

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

> **Note:** Accept sum of three roots set to $-\frac{36}{m}$.
>
> Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$$3\text{i}-3\text{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$$

substitute their r into product of roots ***(M1)***

$(3\text{i})(-3\text{i})\left(\frac{36}{m}\right)=4m$ OR $(z^2+9)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

continued…

Question 7 continued

OR

attempt to equate product of three roots to $4m$ ***(M1)***

> **Note:** Accept product of three roots set to $-4m$.
> Award ***M0*** for stating product of roots is $\pm 4m$.

$(3\mathrm{i})(-3\mathrm{i})\times r = 4m\left(\Rightarrow r = \frac{4m}{9}\right)$

substitute their r into sum of roots ***(M1)***

$3\mathrm{i}-3\mathrm{i}+\frac{4m}{9}=\frac{36}{m}$ OR $\left(z^2+9\right)\left(\frac{4m}{9}-z\right)$

$\frac{4m}{9}=\frac{36}{m}$

THEN

$m=9$ ***(A1)***

third root is 4 ***A1***

[6 marks]

Question 7 continued

METHOD 2

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

recognition that the other factor is $(z+3\text{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\text{i})(z+3\text{i})(r-z)$ OR $(z-3\text{i})(z+3\text{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

Note:	Accept any attempt at long division of $P(z)$ by z^2+9. Award ***M0*** for stating other factor is $(z+3\text{i})$ or obtaining z^2+9 with no further working.

attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

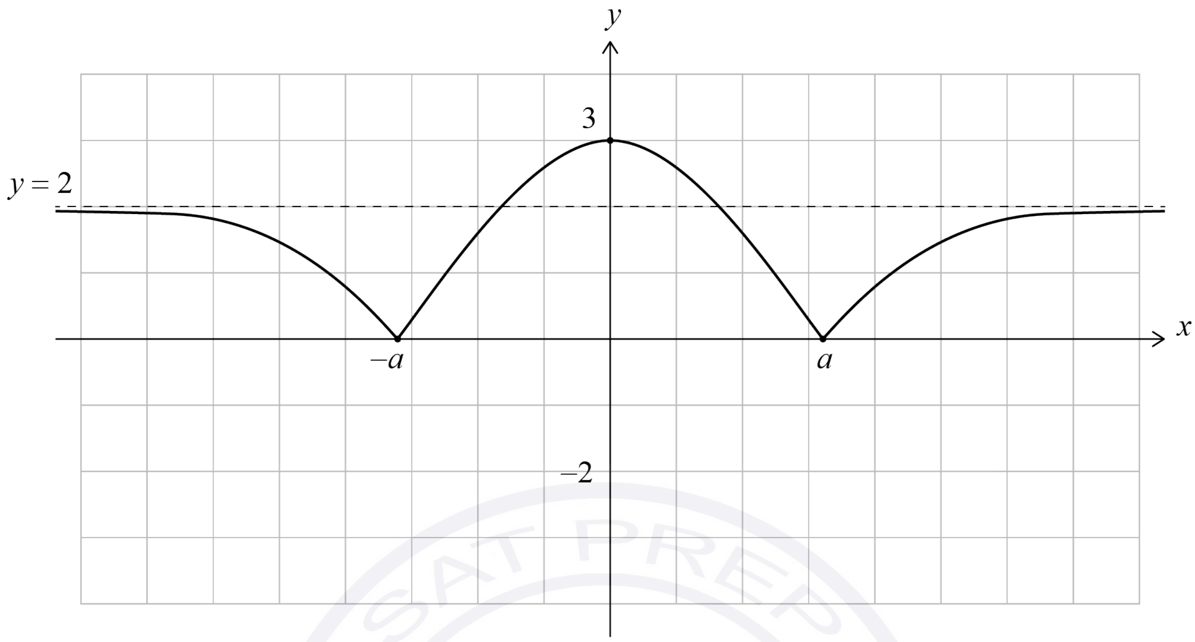
$m=9$ ***(A1)***

third root is 4 ***A1***

Note:	Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\text{i})(z+3\text{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

8. (a) attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:
A1 for correct asymptotic behaviour at $y = 2$ (either side)
A1 for correctly reflected RHS of the graph in the *y*-axis with smooth maximum at (0, 3).
A1 for labelled *x*-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) $k = 0$ ***A1***

$4 \le k < 9$ ***A2***

Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.
Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the *y*-axis.

[3 marks]

Total [7 marks]

9. METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2h}{2}-\frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi\left(r^2-\frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2h}{2}-\frac{h^3}{24}\right)-\pi r^2h+\frac{\pi h^3}{4}=\pi$ OR $\frac{h^3}{4}-\frac{h^3}{12}=1$ (or equivalent)

OR

recognition that the volume of the ring is $\pi\int\left(\left(r^2-y^2\right)-\left(r^2-\frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8}-\frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=\pi$ OR $2\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=1$ (or equivalent)

THEN

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $= \pi\int\left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi\left(r^2 - \frac{h^2}{4}\right)h \quad \left(= \frac{4}{3}\pi r^3 - \pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}\right) + \pi r^2 h - \frac{\pi h^3}{4} = \frac{4}{3}\pi r^3 - \pi$ OR $\frac{h^3}{12} - \frac{h^3}{4} = -1$ (or equivalent)

$h = \sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi\int x\sqrt{r^2 - x^2}\,\mathrm{d}x$

correct limits *r* and $\sqrt{r^2 - \frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$-\frac{1}{3}\left(0 - \left(r^2 - \left(r^2 - \frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation ***(A1)***

$2 \times \frac{-2\pi}{3}\left(0 - \left(r^2 - \left(r^2 - \frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right) = \pi$ OR $2\left(\frac{2\pi}{3} \times \frac{h^3}{8}\right) = \pi$

$h = \sqrt[3]{6}$ ***A1***

[7 marks]

Section B

10. (a) (i) recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

(ii) **METHOD 1**

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[4 marks]

(b) recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(u_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

continued…

Question 10 continued

(c) **EITHER**

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

(d) recognition that $v_2 r^2 = v_4$ OR $\left(v_3\right)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \quad \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

Total [14 marks]

11. (a) $L = \text{AC} + \text{CB}$

$\dfrac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left(\Rightarrow \text{AC} = \dfrac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \dfrac{3}{4}\sec\alpha \right)$ ***A1***

$\dfrac{6}{\text{CB}} = \sin\alpha \left(\Rightarrow \text{CB} = \dfrac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\operatorname{cosec}\alpha \right)$ ***A1***

so $L = \dfrac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$ ***AG***

[2 marks]

(b) (i) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{3}{4}\sec\alpha\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \dfrac{6\cos\alpha}{\sin^2\alpha}$ OR $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \dfrac{6}{\sin\alpha\tan\alpha} \left(= \dfrac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\dfrac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[5 marks]

continued…

Question 11 continued

(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha+\frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha+6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ (or equivalent) ***A1A1***

> **Note:** Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha+\frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha+6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{3}{4}\sec\alpha\tan^2\alpha+\frac{3}{4}\sec^3\alpha+6\operatorname{cosec}\alpha\cot^2\alpha+6\operatorname{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

> **Note:** Award ***M0*** for $\alpha=\arctan 2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\alpha=\sqrt{5}$ OR $\operatorname{cosec}\alpha=\frac{\sqrt{5}}{2}$ OR $\cot\alpha=\frac{1}{2}$ OR $\cos\alpha=\frac{1}{\sqrt{5}}$ OR $\sin\alpha=\frac{2}{\sqrt{5}}$

> **Note:** ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right)+\frac{3}{4}\left(\sqrt{5}\right)^3+6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2+6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent) ***A2***

$$\frac{12\sqrt{5}}{4}+\frac{15\sqrt{5}}{4}+\frac{3\sqrt{5}}{4}+\frac{15\sqrt{5}}{4}$$

> **Note:** Award ***A1*** for only two or three correct terms.
> Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{45}{4}\sqrt{5}$ ***AG***

[7 marks]

continued…

Question 11 continued

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

(ii) $\left(L_{\min} =\right)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

(e) $\left(11.25 =\right)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [19 marks]

12. (a) $2t + 1 \times 0 + 0 \times (3+t) \quad (= 2t)$ (seen anywhere) ***(A1)***

one correct magnitude $\sqrt{1^2 + 1^2 + 0^2}, \sqrt{(2t)^2 + (3+t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t = \sqrt{2} \times \sqrt{(2t)^2 + (3+t)^2} \times \cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3} = \frac{2t}{\sqrt{2} \times \sqrt{5t^2 + 6t + 9}}$

$4t = \sqrt{2\left(4t^2 + 9 + 6t + t^2\right)}$ OR $\frac{1}{2} = \frac{2t}{\sqrt{2\left(5t^2 + 6t + 9\right)}}$ (or equivalent) ***A1***

$4t = \sqrt{10t^2 + 12t + 18}$ ***AG***

[4 marks]

(b) correct quadratic equation ***A1***

$16t^2 = 10t^2 + 12t + 18$, $6t^2 - 12t - 18 = 0$, $t^2 - 2t - 3 = 0$

valid attempt to solve their quadratic set =0 ***(M1)***

$(t+1)(t-3)$ OR $\frac{12 \pm \sqrt{(-12)^2 - 4 \times 6 \times (-18)}}{12}$ OR $(t-1)^2 - 4$ ***(A1)***

$t = 3$ ***A1***

Note: Award ***A0*** if additional answer(s) given.

[4 marks]

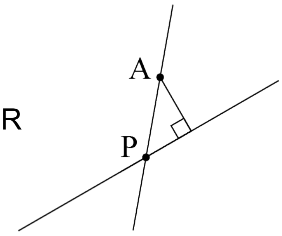
continued…

Question 12 continued

(c) **METHOD 1**

recognizing shortest distance from A is perpendicular to L_1 ***(M1)***

OR



$\sin\frac{\pi}{3} = \frac{x}{\left|\overrightarrow{PA}\right|}$

$\left|\overrightarrow{PA}\right| = \sqrt{6^2 + 6^2}$ $\left(= \sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

$\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{72}}$ ***(A1)***

$x = \frac{\sqrt{216}}{2}$ $\left(= \sqrt{54}\ ,\ 3\sqrt{6}\right)$

shortest distance is $\frac{\sqrt{216}}{2}$ $\left(= \sqrt{54}\ ,\ 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 2

recognition that the distance required is $\dfrac{\left|\boldsymbol{v} \times \overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$= \dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}6\\0\\6\end{pmatrix}\right|$ ***(A1)***

$= \dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right|$ ***(A1)***

shortest distance is $\sqrt{54}\ \left(= 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

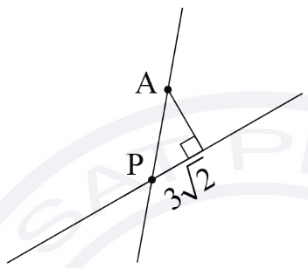
Question 12 continued

METHOD 3

recognition that the base of the triangle is $\dfrac{\left|\boldsymbol{v}\cdot\overrightarrow{\text{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$\dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$

$=\dfrac{6}{\sqrt{2}}\left(=3\sqrt{2}\right)$ OR



(A1)

$\left|\overrightarrow{\text{PA}}\right|=\sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

Note: The value of $\left|\overrightarrow{\text{PA}}\right|=\sqrt{6^2+6^2}$ may be seen as part of the working of their shortest distance, $d=\sqrt{\left|\overrightarrow{\text{PA}}\right|^2-b^2}=\sqrt{\left(\sqrt{72}\right)^2-\left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 4

Let B be a general point on L_1 $(\lambda, 8+\lambda, -3)$ such that AB is perpendicular to L_1

attempt to find vector $\vec{\mathrm{AB}}$ OR $\left|\vec{\mathrm{AB}}\right|$ (the shortest distance from A to L_1) ***(M1)***

$$\vec{\mathrm{AB}} = \vec{\mathrm{OB}} - \vec{\mathrm{OA}} = \vec{\mathrm{OP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \vec{\mathrm{OA}}\left(=\begin{pmatrix}0\\8\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}6\\8\\3\end{pmatrix}=\vec{\mathrm{AP}}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\quad(\lambda\in\mathbb{R})$$

$$\vec{\mathrm{AB}} = \begin{pmatrix}-6\\0\\-6\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ OR } \left|\vec{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2+(8+\lambda-8)^2+(-3-3)^2}$$ ***A1***

$$\left|\vec{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2+\lambda^2+(-6)^2}\left(=\sqrt{2\lambda^2-12\lambda+72}\right)$$

EITHER

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\vec{\mathrm{AB}}\right|^2\right)=0\Rightarrow 4\lambda-12=0\Rightarrow\lambda=3$$ ***A1***

OR

$\left|\vec{\mathrm{AB}}\right| = \sqrt{2(\lambda-3)^2+54}$ to obtain $\lambda = 3$ ***A1***

OR

$$\begin{pmatrix}-6+\lambda\\\lambda\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=0\Rightarrow-6+\lambda+\lambda=0\Rightarrow\lambda=3$$ ***A1***

THEN

shortest distance is $\sqrt{54}\left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

(d) attempt to find the vector product of two direction vectors ***(M1)***

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$$

$\boldsymbol{n} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (or any scalar multiple of this) (accept $\boldsymbol{n} =< 1, -1, -1 >$ or equivalent) ***A1***

Note: Award ***A0*** for a final answer given in coordinate form.

[2 marks]

continued…

Question 12 continued

(e) substituting their x into volume formula and equating ***(M1)***

$$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$$

$h = 5\sqrt{3}$ (seen anywhere) ***A1***

recognition that the position vector of vertex is given by $\vec{\mathrm{OA}} + \mu \boldsymbol{n}$ OR $\vec{\mathrm{OA}} + h \times \hat{\boldsymbol{n}}$ ***(M1)***

$$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \text{ OR } \left(6+\mu, 8-\mu, 3-\mu\right)$$

EITHER

recognition that $\mu|\boldsymbol{n}| = h$ (where μ is a parameter) ***(M1)***

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) ***(A1)***

OR

attempt to find cone's height vector $h \times \hat{\boldsymbol{n}}$ ***(M1)***

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$ ***(A1)***

$$5\sqrt{3} \times \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

THEN

$$= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm 5 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} \right)$$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) ***A1A1***

Note: Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for $\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\vec{\mathrm{PA}}\right|$ from part (c).

[7 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2022

Mathematics: analysis and approaches

Higher level

Paper 1

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. $g'(x) = 2x\mathrm{e}^{x^2+1}$ ***(A2)***

substitute $x = -1$ into **their** derivative ***(M1)***

$g'(-1) = -2\mathrm{e}^2$ ***A1***

Note: Award ***A0M0A0*** in cases where candidate's incorrect derivative is $g'(x) = \mathrm{e}^{x^2+1}$.

[4 marks]

2.

(a) (i) attempt to find midpoint of A and B ***(M1)***

centre $(-1,3,-2)$ (accept vector notation and/or missing brackets) ***A1***

(ii) attempt to find AB or half of AB or distance between the centre and A (or B) ***(M1)***

$\frac{\sqrt{4^2+2^2+4^2}}{2}$ or $\sqrt{2^2+1^2+2^2}$

$=3$ ***A1***

[4 marks]

(b) attempt to find the distance between their centre and V
(the perpendicular height of the cone) ***(M1)***

$\sqrt{0^2+4^2+2^2}$ OR $\sqrt{(\text{their slant height})^2-(\text{their radius})^2}$

$=\sqrt{20}\left(=2\sqrt{5}\right)$ ***(A1)***

Volume $=\frac{1}{3}\pi 3^2\sqrt{20}$

$=3\pi\sqrt{20}\left(=6\pi\sqrt{5}\right)$ ***A1***

[3 marks]

Total [7 marks]

3. (a)

Note: Award a maximum of ***M1A0A0*** if the candidate manipulates both sides of the equation (such as moving terms from one side to the other).

METHOD 1 (working with LHS)

attempting to expand $\left(a^2-1\right)^2$ (do not accept a^4+1 or a^4-1) ***(M1)***

$\text{LHS}=a^2+\frac{a^4-2a^2+1}{4}$ or $\frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$=\frac{a^4+2a^2+1}{4}$ ***A1***

$=\left(\frac{a^2+1}{2}\right)^2$ $(=\text{RHS})$ ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

METHOD 2 (working with RHS)

attempting to expand $\left(a^2+1\right)^2$ ***(M1)***

$\text{RHS}=\frac{a^4+2a^2+1}{4}$

$=\frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$=a^2+\frac{a^4-2a^2+1}{4}$ ***A1***

$=a^2+\left(\frac{a^2-1}{2}\right)^2$ (=LHS) ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

[3 marks]

continued…

Question 3 continued

(b) recognise base and height as a and $\left(\dfrac{a^2-1}{2}\right)$ (may be seen in diagram) ***(M1)***

correct substitution into triangle area formula ***A1***

$$\text{Area} = \frac{a}{2}\left(\frac{a^2-1}{2}\right) \text{ (or equivalent) } \left(= \frac{a\left(a^2-1\right)}{4} = \frac{a^3-a}{4}\right)$$

[2 marks]

Total [5 marks]

4. recognizing need to integrate *(M1)*

$\int \frac{6x}{x^2+1}\mathrm{d}x$ OR $u = x^2+1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\mathrm{d}x$ *(A1)*

$= 3\ln\left(x^2+1\right)(+c)$ or $3\ln u(+c)$ ***A1***

correct substitution of $x=1$ and $f(x)=5$ or $x=1$ and $u=2$ into equation

using **their** integrated expression (must involve *c*) *(M1)*

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2+1\right) + 5 - 3\ln 2$ $\left(= 3\ln\left(x^2+1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$ (or equivalent) ***A1***

Note: Accept the use of the modulus sign in working and the final answer.

[5 marks]

5. (a) product of roots $=80$ ***(A1)***

$3-\mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3+\mathrm{i})(3-\mathrm{i})\alpha^3=80 \Rightarrow 10\alpha^3=80$

$\alpha=2$ ***A1***

[4 marks]

(b) **METHOD 1**

sum of roots $=-p$ ***(A1)***

$-p=3+\mathrm{i}+3-\mathrm{i}+2+4$ ***(M1)***

Note: Accept $p=3+\mathrm{i}+3-\mathrm{i}+2+4$ for ***(M1)***

$p=-12$ ***A1***

METHOD 2

$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))(z-2)(z-4)$ ***(M1)***

$((z-3)-i)((z-3)+i)(z-2)(z-4)$ ***(A1)***

$(z^2-6z+10)(z^2-6z+8)=z^4-12z^3+....$

$p=-12$ ***A1***

[3 marks]

Total [7 marks]

6. (a) $\mathrm{P}(A \cap B) = 0.24$ ***A1***

[1 mark]

(b) $\mathrm{P}(A \cup B) = 1.1 - \mathrm{P}(A \cap B)$ ***(A1)***

$(0 \le)\mathrm{P}(A \cup B) \le 1$ ***(M1)***

Note: This may be conveyed in a clearly labelled diagram or written explanation where $\mathrm{P}(A \cup B) = 1$

the minimum value of $\mathrm{P}(A \cap B)$ is 0.1 ***A1***

[3 marks]

(c) A is a subset of B (so $\mathrm{P}(A \cap B) = \mathrm{P}(A)$). ***R1***

Note: This may be conveyed in a clearly labelled diagram where A is completely inside B, or in a written explanation indicating that $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

so the maximum value of $\mathrm{P}(A \cap B)$ is 0.3 ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [6 marks]

7. attempt at implicit differentiation, including use of the product rule ***(M1)***

EITHER

$$\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2+\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}=8x$$ ***A1A1A1***

Note: Award ***A1*** for each of $\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2$, $\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

OR

$$x^2y^2+y^4=4x^2$$

$$2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=8x$$ ***A1A1A1***

Note: Award ***A1*** for each of $2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

THEN

at a local maximum or minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***(M1)***

$$2xy^2=8x$$

$x=0 \text{ or } y^2=4\left(\Rightarrow y=\pm 2\right)$ ***A1***

Note: Award ***A0*** for $x=0 \text{ or } y=2$

since $x>0$ and $-2<y<2$ there are no solutions ***R1***

hence there are no local maximum or minimum points ***AG***

[7 marks]

8. (a)

$a = \frac{\pi}{2}$ ***A2***

Note: For sinusoidal graph through the origin seen with incorrect a, or use of horizontal line test with incorrect a, award ***A1A0***

(b) $a = \pi$ ***A1***

(c)

sketch showing sinusoidal shape decreasing as it crosses the y-axis (below or above the origin) ***(A1)***

$a = k - \pi$ ***A1***

[5 marks]

9. (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y^2-2x^2}{xy}$

let $y=vx$

$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=v+x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\Rightarrow v+x\dfrac{\mathrm{d}v}{\mathrm{d}x}=\dfrac{v^2x^2-2x^2}{vx^2}$ ***(M1)***

$\Rightarrow x\dfrac{\mathrm{d}v}{\mathrm{d}x}=\dfrac{-2}{v}$ ***(A1)***

$\Rightarrow \int v\mathrm{d}v=-\int\dfrac{2}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \dfrac{v^2}{2}=-2\ln|x|(+c)$ ***A1***

Note: Condone the absence of the modulus sign up to this point.

$\Rightarrow \dfrac{y^2}{2x^2}=-2\ln|x|+c$ ***A1***

attempt to substitute $x=1, y=2$ into their integrated expression to find c ***M1***

$\Rightarrow 2=-2\ln|1|+c\Rightarrow c=2$

$\Rightarrow \dfrac{y^2}{2x^2}=-2\ln|x|+2$

$\Rightarrow y^2=2x^2\left(-2\ln|x|+2\right)\left(=4x^2\left(1-\ln|x|\right)\right)$ ***A1***

[8 marks]

continued…

Question 9 continued

(b) attempt to set $\frac{dy}{dx} = 0$ in the differential equation ***(M1)***

$y = \sqrt{2}x$ and $y = -\sqrt{2}x$ or $m = \pm\sqrt{2}$ ***A1***

[2 marks]

Total [10 marks]

Section B

10. (a) $\cos^2 x - 3\sin^2 x = 0$

valid attempt to reduce equation to one involving one trigonometric function ***(M1)***

$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3}$ OR $1 - \sin^2 x - 3\sin^2 x = 0$ OR $\cos^2 x - 3\left(1 - \cos^2 x\right) = 0$

OR $\cos 2x - 1 + \cos 2x = 0$

correct equation ***(A1)***

$\tan^2 x = \frac{1}{3}$ OR $\cos^2 x = \frac{3}{4}$ OR $\sin^2 x = \frac{1}{4}$ OR $\cos 2x = \frac{1}{2}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ OR $\cos x = \pm\frac{\sqrt{3}}{2}$ OR $\sin x = (\pm)\frac{1}{2}$ OR $2x = \frac{\pi}{3}\left(, \frac{5\pi}{3}\right)$ ***(A1)***

$x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$ ***A1A1***

Note: Award ***M1A1A0A1A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = \frac{\pi}{6}$.

Award ***M1A1A0A0A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = 30°$.

Award ***M1A1A1A1A0*** for candidates who give both answers in degrees.

Award ***M1A1A1A1A0*** for candidates who give both correct answers in radians, but who include additional solutions outside the domain.

Award a maximum of ***M1A0A0A1A1*** for correct answers with no working.

[5 marks]

continued…

Question 10 continued

(b) (i) attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) ***(M1)***

$f'(x) = -2\cos x \sin x - 6\sin x \cos x \left(= -8\sin x \cos x = -4\sin 2x\right)$ ***A1***

(ii) valid attempt to solve their $f'(x) = 0$ ***(M1)***

at least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0$, $x = \dfrac{\pi}{2}$, $x = \pi$

Note: Accept additional correct solutions outside the domain. Award ***A0*** if any additional incorrect solutions are given.

correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0,1)$, $(\pi,1)$, $\left(\dfrac{\pi}{2}, -3\right)$

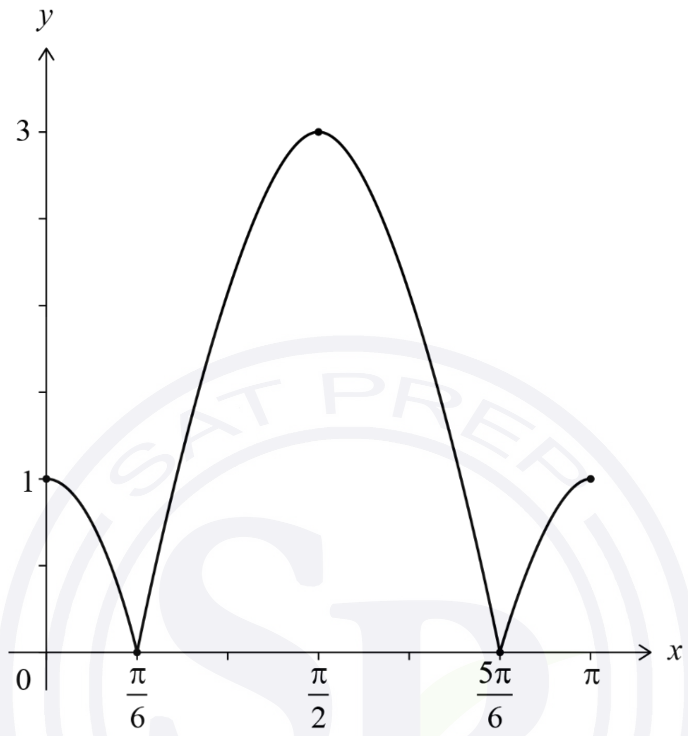
Note: Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given. **Note:** If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as ***M1A0A0A1A0.***

[7 marks]

continued…

Question 10 continued

(c)



attempt to reflect the negative part of the graph of f in the x-axis ***M1***

endpoints have coordinates $(0,1)$, $(\pi,1)$ ***A1***

smooth maximum at $\left(\frac{\pi}{2},3\right)$ ***A1***

sharp points (cusps) at x-intercepts $\frac{\pi}{6}$, $\frac{5\pi}{6}$ ***A1***

[4 marks]

continued…

Question 10 continued

(d) considers points of intersection of $y=\left|f\left(x\right)\right|$ and $y=1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x-3\sin^2 x\right)=1$ or $-\left(1-4\sin^2 x\right)=1$ or $-\left(4\cos^2 x-3\right)=1$ or $-\left(2\cos 2x-1\right)=1$

$\tan^2 x=1$ or $\sin^2 x=\frac{1}{2}$ or $\cos^2 x=\frac{1}{2}$ or $\cos 2x=0$ ***(A1)***

$x=\frac{\pi}{4},\frac{3\pi}{4}$ ***(A1)***

For $\left|f\left(x\right)\right|>1$

$\frac{\pi}{4}<x<\frac{3\pi}{4}$ ***A1***

[4 marks]

Total [20 marks]

11. (a) (i) 5^3 ***(A1)***

$= 125$ ***A1***

(ii) ${}^5P_3 = 5 \times 4 \times 3$ ***(A1)***

$= 60$ ***A1***

[4 marks]

(b) (i) **METHOD 1**

$x^2 + 3x + 2 = (x+1)(x+2)$ ***(A1)***

correct use of factor theorem for at least one of their factors ***(M1)***

$P(-1) = 0$ or $P(-2) = 0$

attempt to find two equations in a, b and c ***(M1)***

$(-1)^3 + a(-1)^2 + b(-1) + c = 0 (\Rightarrow -1 + a - b + c = 0)$

$(-2)^3 + a(-2)^2 + b(-2) + c = 0$

$-8 + 4a - 2b + c = 0$ and $-1 + a - b + c = 0$ ***A1***

attempt to combine their two equations in $-8 + 4a - 2b + c = 0$ to eliminate c ***(M1)***

$b = 3a - 7$ ***A1***

Note: Award at most ***A1M1M1A0M1A0*** for $b = -3a - 7$ from $P(1) = P(2) = 0$

continued…

Question 11 continued

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d)$ ***(M1)***

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ ***(A1)***

attempt to compare coefficients of x^2 and x ***(M1)***

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d ***(M1)***

$\Rightarrow b = 3a - 7$ ***A1***

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ ***M1***

$$\frac{x^3 + ax^2 + bx + c}{x^2 + 3x + 2} = (x + a - 3) + \frac{(-3a + b + 7)x + (c - 2a + 6)}{x^2 + 3x + 2}$$ ***A1A1A1***

Note: Award ***A1*** *for* $x + a - 3$, ***A1*** *for* $(-3a + b + 7)x$ and ***A1*** *for* $c - 2a + 6$

recognition that, if $(x^2 + 3x + 2)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ ***(M1)***

leading to $b = 3a - 7$ ***A1***

continued…

Question 11 continued

METHOD 4

$x^2+3x+2=(x+1)(x+2)$ ***(A1)***

attempt to use Vieta's formulae for a cubic with roots –1, –2 and "*p*" ***(M1)***

$(-1)+(-2)+p=-a\left(\Rightarrow p=3-a\right)$ ***A1***

$(-1)(-2)+(-1)p+(-2)p=b$ ***A1***

Attempt to eliminate "*p*" ***(M1)***

$2-(3-a)-2(3-a)=b$

$b=3a-7$ ***A1***

Note: Award at most ***A1M1A0A0M1A0*** for $b=-3a-7$ from roots 1, 2 and "*p*"

(ii) **METHOD 1**

$a=1,2,5$ lead to invalid values for b ***R1***

$a=3,b=2\Rightarrow c=0$ so not possible ***R1***

so $a=4,b=5,c=2$ is the only solution ***AG***

METHOD 2

$c=2a-6$ ***R1***

correctly argues $a=4$ is the only possibility ***R1***

so $a=4,b=5,c=2$ is the only solution ***AG***

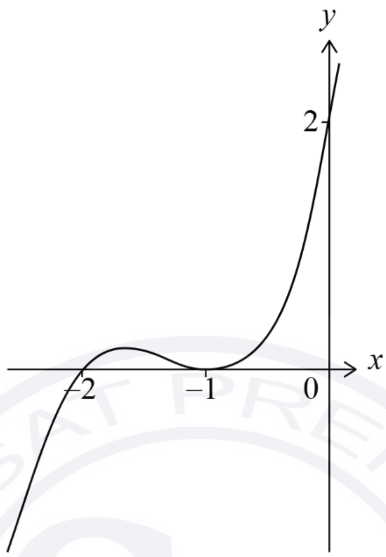
(iii) $x^3+4x^2+5x+2=\left(x^2+3x+2\right)(x+1)$

$=(x+2)(x+1)(x+1)$ ***A1***

continued…

Question 11 continued

(iv)



positive cubic shape with y-intercept at $(0, 2)$ ***A1***

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ ***A1***

local minimum point at $(-1, 0)$ ***A1***

Note: Accept answers from an approach based on calculus.

[12 marks]

Total [16 marks]

12. (a) (i) $z_0 = 1 + \mathrm{i}$ ***(A1)***

$\arg(z_0) = \arctan(1) = \frac{\pi}{4} = 45°$ ***A1***

Note: Accept any of these three forms, including an answer marked on an Argand diagram.

(ii) $\arg(z_n) = \arctan\left(\frac{1}{n^2+n+1}\right)$ ***A1***

[3 marks]

(b) (i) attempt to use the compound angle formula for tan ***M1***

$$\tan(\arctan(a) + \arctan(b)) = \frac{\tan(\arctan(a)) + \tan(\arctan(b))}{1 - \tan(\arctan(a))\tan(\arctan(b))}$$

$= \frac{a+b}{1-ab}$ ***A1***

$\Rightarrow \arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ ***AG***

(ii) **METHOD 1**

$\arg(w_1) = \arg(z_0 z_1) = \arg(z_0) + \arg(z_1)$ ***M1***

$= \arctan(1) + \arctan\left(\frac{1}{3}\right)$ ***(A1)***

$= \arctan\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$ ***A1***

$= \arctan(2)$ ***AG***

continued…

Question 12 continued

METHOD 2

$w_1 = z_0 z_1 = (1+\mathrm{i})(3+\mathrm{i})$ ***(M1)***

$= 2 + 4\mathrm{i}$ ***A1***

$\arg(w_1) = \arctan\left(\frac{4}{2}\right)$ or labelled Argand diagram ***A1***

$= \arctan(2)$ ***AG***

[5 marks]

continued…

Question 12 continued

(c) let $n = 0$

$$\text{LHS} = \arg(w_0) = \arg(z_0) = \arctan(1)\left(= \frac{\pi}{4}\right)$$

$$\text{RHS} = \arctan(1)\left(= \frac{\pi}{4}\right)$$ so LHS = RHS **R1**

> **Note:** Award ***R0*** for not starting at $n = 0$, for example by referring to the result in (b) (ii) for $n = 1$. Award subsequent marks.

assume true for $n = k$, (so $\arg(w_k) = \arctan(k+1)$) **M1**

> **Note:** Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

$$\arg(w_{k+1})$$

$$= \arg(w_k z_{k+1})\left(= \arg(w_k) + \arg(z_{k+1})\right)$$ ***(M1)***

$$= \arctan(k+1) + \arctan\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$ ***A1***

$$= \arctan\left(\frac{(k+1) + \left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}{1 - (k+1)\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}\right)$$ ***M1***

$$= \arctan\left(\frac{(k+1) + \left(\frac{1}{k^2 + 3k + 3}\right)}{1 - (k+1)\left(\frac{1}{k^2 + 3k + 3}\right)}\right)$$ ***(A1)***

$$= \arctan\left(\frac{(k+1)(k^2 + 3k + 3) + 1}{(k^2 + 3k + 3) - (k+1)}\right)$$

continued…

Question 12 continued

$$=\arctan\left(\frac{k^3+4k^2+6k+4}{k^2+2k+2}\right)$$ ***A1***

$$=\arctan\left(\frac{(k+2)(k^2+2k+2)}{k^2+2k+2}\right)$$ ***A1***

$$=\arctan(k+2)\left(=\arctan((k+1)+1)\right)$$ ***A1***

since true for $n=0$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n \in \mathbb{N}$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** four of the previous marks must have been awarded.

[10 marks]

Total [18 marks]

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) $u_1 = 12$ ***A1***

[1 mark]

(b) $15 - 3n = -33$ ***(A1)***

$n = 16$ ***A1***

[2 marks]

(c) valid approach to find d ***(M1)***

$u_2 - u_1 = 9 - 12$ OR recognize gradient is -3 OR attempts to solve $-33 = 12 + 15d$

$d = -3$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $(n-1)+n+(n+1)$ ***(A1)***

$=3n$ ***A1***

which is always divisible by 3 ***AG***

[2 marks]

(b) $(n-1)^2+n^2+(n+1)^2$ $(=n^2-2n+1+n^2+n^2+2n+1)$ ***A1***

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept n^2-1 or n^2+1) ***(M1)***

$=3n^2+2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by 3 and so $3n^2+2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by 3 ***AG***

[4 marks]

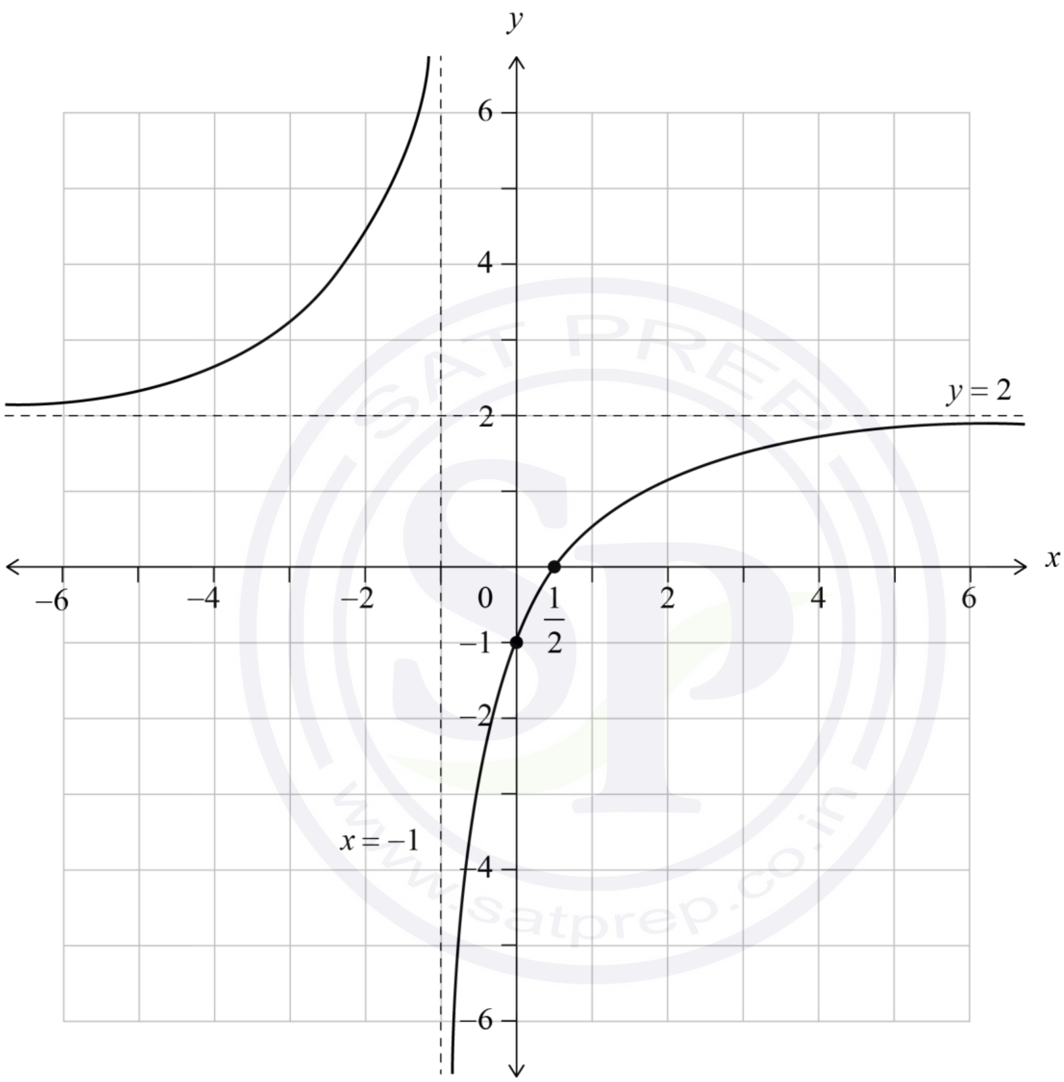
Total [6 marks]

3. (a) (i) $x = -1$ ***A1***

(ii) $y = 2$ ***A1***

[2 marks]

(b)



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x=\frac{1}{2}$ and $y=-1$ ***A1A1***

[3 marks]

continued…

Question 3 continued

(c) $x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2}, \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) **EITHER**

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x|-1=0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

Total [8 marks]

4. determines $\frac{\pi}{4}$ (or $45°$) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

> **Note:** Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4},\frac{7\pi}{4}(,...)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\ \left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

[5 marks]

5. (a) **EITHER**

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}^7C_2\, x^5 1^2$ OR $b = {}^7C_2$ OR ${}^7C_5$

$$b = \frac{7!}{2!5!} \left(= \frac{7!}{2!(7-2)!} \right)$$

correct working ***A1***

$$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1} \quad \text{OR} \quad \frac{7\times6}{2!} \quad \text{OR} \quad \frac{42}{2}$$

OR

lists terms from row 7 of Pascal's triangle ***(M1)***

$1, 7, 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) $a = 7$ ***(A1)***

correct equation ***A1***

$$21x^5 = \frac{ax^6 + 35x^4}{2} \quad \text{OR} \quad 21x^5 = \frac{7x^6 + 35x^4}{2}$$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic ***(M1)***

$$(x-1)(x-5) = 0 \quad \text{OR} \quad x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$$

$x = 1,\ x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

[5 marks]

Total [7 marks]

6. (a) attempts to replace x with $-x$ ***M1***

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-(-x)^2}\ (= -f(x))$ ***A1***

> **Note:** Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.
>
> **Note:** Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by $180°$ about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$f'(x) = x \times \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \times 1 \left(= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f'(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

> **Note:** Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x) = 0$ roots.

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

> **Note:** Award ***A1*** for $-\frac{1}{2} \le y \le \frac{1}{2}$.

[6 marks]

Total [8 marks]

7. METHOD 1

$u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \ \mathrm{d}x$ ***(A1)***

attempts to express the integral in terms of u ***M1***

$\int_1^2 u^{n-1} \mathrm{d}u$ ***A1***

$= \frac{1}{n}\left[u^n\right]_1^2 \quad \left(= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ ***A1***

Note: Condone the absence of or incorrect limits up to this point.

$= \frac{2^n - 1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

Note: Award ***M1*** for correct substitution of **<u>their</u>** limits for u into their antiderivative for u (or given limits for x into their antiderivative for x).

METHOD 2

$\int \sec^n x \tan x \ \mathrm{d}x = \int \sec^{n-1} x \sec x \tan x \ \mathrm{d}x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}$ ***A2***

Note: Award ***A2*** if the limits are not stated.

$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$ ***M1***

Note: Award ***M1*** for correct substitution into their antiderivative.

$= \frac{2^n - 1}{n}$ ***A1***

[6 marks]

8. let m be the median

EITHER

attempts to find the area of the required triangle ***M1***

base is $(m-a)$ ***(A1)***

and height is $\frac{2}{(b-a)(c-a)}(m-a)$

area $=\frac{1}{2}(m-a)\times\frac{2}{(b-a)(c-a)}(m-a)\ \left(=\frac{(m-a)^2}{(b-a)(c-a)}\right)$ ***A1***

OR

attempts to integrate the correct function ***M1***

$$\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$$

$= \frac{2}{(b-a)(c-a)}\left[\frac{1}{2}(x-a)^2\right]_a^m$ OR $\frac{2}{(b-a)(c-a)}\left[\frac{x^2}{2}-ax\right]_a^m$ ***A1A1***

Note: Award ***A1*** for correct integration and ***A1*** for correct limits.

THEN

sets up (their) $\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$ or area $=\frac{1}{2}$ ***M1***

Note: Award ***M0A0A0M1A0A0*** if candidates conclude that $m>c$ and set up their area or sum of integrals $=\frac{1}{2}$.

$$\frac{(m-a)^2}{(b-a)(c-a)}=\frac{1}{2}$$

$m=a\pm\sqrt{\frac{(b-a)(c-a)}{2}}$ ***(A1)***

as $m>a$, rejects $m=a-\sqrt{\frac{(b-a)(c-a)}{2}}$

so $m=a+\sqrt{\frac{(b-a)(c-a)}{2}}$ ***A1***

[6 marks]

9. METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

> **Note:** Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.
> Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.
>
> **Note:** Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"
>
> **Note:** Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

> **Note:** Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

> **Note:** Accept '$\Rightarrow 1$ is even which gives a contradiction'.

continued…

Question 9 continued

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ and so α cannot be an integer ***R1***

Note: Accept '$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2\left(\alpha^2 + 3\right)}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\left(\alpha^2 + 3\right)} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

continued…

Question 9 continued

METHOD 2

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R} \Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

Section B

10. (a) uses $\sum \mathrm{P}(X=x)=1$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from summing to 1 ***A1***

$p+0.3+q+0.1=1$ OR $p+q=0.6$ (or equivalent)

uses $\mathrm{E}(X)=2$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from $\mathrm{E}(X)=2$ ***A1***

$p+0.6+3q+0.4=2$ OR $p+3q=1$ (or equivalent)

Note: The marks for using $\sum \mathrm{P}(X=x)=1$ and the marks for using $\mathrm{E}(X)=2$ may be awarded independently of each other.

evidence of correctly solving these equations simultaneously ***A1***

for example, $2q=0.4 \Rightarrow q=0.2$ or $p+3\times(0.6-p)=1 \Rightarrow p=0.4$

so $p=0.4$ and $q=0.2$ ***AG***

[5 marks]

(b) valid approach ***(M1)***

$\mathrm{P}(X>2)=\mathrm{P}(X=3)+\mathrm{P}(X=4)$ OR $\mathrm{P}(X>2)=1-\mathrm{P}(X=1)-\mathrm{P}(X=2)$

$=0.3$ ***A1***

[2 marks]

continued…

Question 10 continued

(c) recognises at least one of the valid scores (6, 7, or 8) required to win the game ***(M1)***

Note:	Award ***M0*** if candidate also considers scores other than 6, 7, or 8 (such as 5).

let T represent the score on the last two rolls

a score of 6 is obtained by rolling $(2,4),(4,2)$ or $(3,3)$

$\mathrm{P}(T=6)=2(0.3)(0.1)+(0.2)^2 \quad (=0.1)$ ***A1***

a score of 7 is obtained by rolling $(3,4)$ or $(4,3)$

$\mathrm{P}(T=7)=2(0.2)(0.1) \quad (=0.04)$ ***A1***

a score of 8 is obtained by rolling $(4,4)$

$\mathrm{P}(T=8)=(0.1)^2 \quad (=0.01)$ ***A1***

Note: The above 3 ***A1*** marks are independent of each other.

$\mathrm{P}(\text{Nicky wins})=0.1+0.04+0.01$

$=0.15$ ***A1***

[5 marks]

(d) $3+b=8$ ***(M1)***

$b=5$ ***A1***

[2 marks]

continued…

Question 10 continued

(e) **METHOD 1**

EITHER

$\mathrm{P}(S=5)=\frac{4}{16}$

$\mathrm{P}(S=a+2)=\frac{4}{16}$ ***A1***

$\Rightarrow a+2=5$

OR

$\mathrm{P}(S=6)=\frac{3}{16}$

$\mathrm{P}(S=a+3)=\frac{2}{16}$ and $\mathrm{P}(S=5+1)=\frac{1}{16}$ ***A1***

$\Rightarrow a+3=6$

OR

$\mathrm{P}(S=4)=\frac{3}{16}$

$\mathrm{P}(S=a+1)=\frac{2}{16}$ and $\mathrm{P}(S=1+3)=\frac{1}{16}$ ***A1***

$\Rightarrow a+1=4$

THEN

$\Rightarrow a=3$ ***A1***

Note: Award ***A0A0*** for $a=3$ obtained without working/reasoning/justification.

[2 marks]

continued…

Question 10 continued

METHOD 2

EITHER

correctly lists a relevant part of the sample space ***A1***

for example, $\{S=4\}=\{(3,1),(1,a),(1,a)\}$ or $\{S=5\}=\{(2,a),(2,a),(2,a),(2,a)\}$

or $\{S=6\}=\{(3,a),(3,a),(1,5)\}$

$a+3=6$

OR

eliminates possibilities (exhaustion) for $a<5$

convincingly shows that $a\neq 2,4$ ***A1***

$a\neq 4$, for example, $\mathrm{P}(S=7)=\frac{2}{16}$ from $(2,5),(2,5)$ and so $(3,a),(3,a)\Rightarrow a+3\neq 7$

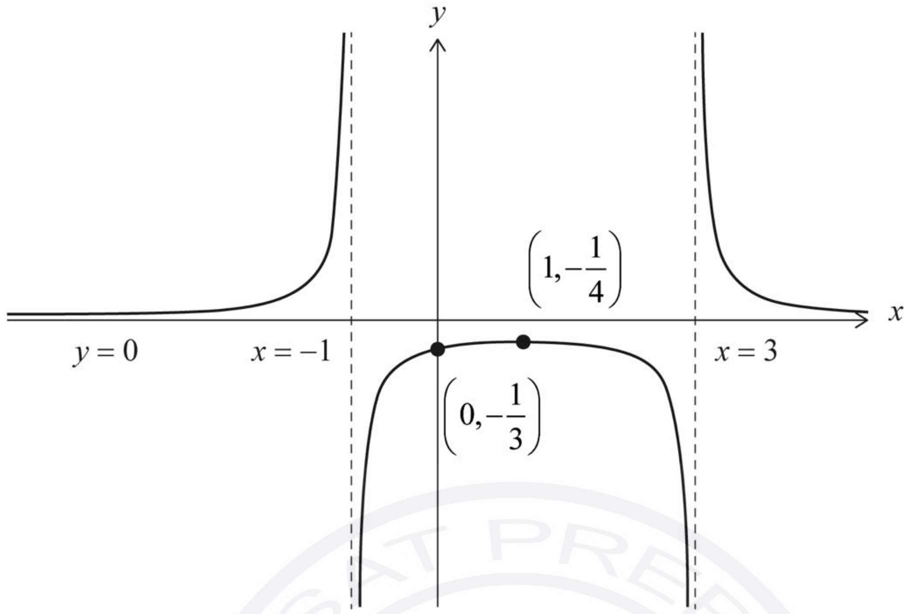
THEN

$\Rightarrow a=3$ ***A1***

[2 marks]

Total [16 marks]

11. (a)



y- intercept $\left(0, -\frac{1}{3}\right)$ ***A1***

Note: Accept an indication of $-\frac{1}{3}$ on the $y-$axis.

vertical asymptotes $x = -1$ and $x = 3$ ***A1***

horizontal asymptote $y = 0$ ***A1***

uses a valid method to find the x- coordinate of the local maximum point ***(M1)***

Note: For example, uses the axis of symmetry or attempts to solve $f'(x) = 0$.

local maximum point $\left(1, -\frac{1}{4}\right)$ ***A1***

Note: Award ***(M1)A0*** for a local maximum point at $x = 1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

[6 marks]

continued…

Question 11 continued

(b) (i) $x=\frac{1}{y^2-2y-3}$ ***M1***

Note: Award ***M1*** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square ***M1***

$y^2-2y-3=(y-1)^2-4$ ***A1***

$x=\frac{1}{(y-1)^2-4}$

$(y-1)^2-4=\frac{1}{x}\ \left((y-1)^2=4+\frac{1}{x}\right)$ ***A1***

$y-1=\pm\sqrt{4+\frac{1}{x}}\ \left(=\pm\sqrt{\frac{4x+1}{x}}\right)$

OR

attempts to solve $xy^2-2xy-3x-1=0$ for y ***M1***

$y=\frac{-(-2x)\pm\sqrt{(-2x)^2+4x(3x+1)}}{2x}$ ***A1***

Note: Award ***A1*** even if **–** (in $\pm$) is missing

$=\frac{2x\pm\sqrt{16x^2+4x}}{2x}$ ***A1***

continued…

Question 11 continued

THEN

$=1\pm\frac{\sqrt{4x^2+x}}{x}$ ***A1***

$y>3$ and hence $y=1-\frac{\sqrt{4x^2+x}}{x}$ is rejected ***R1***

Note: Award ***R1*** for concluding that the expression for y must have the '$+$' sign. The ***R1*** may be awarded earlier for using the condition $x>3$.

$y=1+\frac{\sqrt{4x^2+x}}{x}$

$g^{-1}(x)=1+\frac{\sqrt{4x^2+x}}{x}$ ***AG***

(ii) domain of g^{-1} is $x>0$ ***A1***

[7 marks]

continued…

Question 11 continued

(c) attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2-2a-3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4}$ $\left(\arctan\left(\frac{1}{2(a^2-2a-3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2$ $\left(\frac{1}{(a^2-2a-3)} = 2\right)$

EITHER

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2+2}}{2}$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

OR

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4}$ $\left(= \frac{4 \pm \sqrt{72}}{4}\right)$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

THEN

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

Note: Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}$ $(p = 1,\ q = 1,\ r = 18)$.

[7 marks]

Total [20 marks]

12. (a) $z_2{}^* = r_2 e^{-i\theta}$ ***(A1)***

$z_1 z_2{}^* = r_1 e^{i\alpha} r_2 e^{-i\theta}$ ***A1***

$z_1 z_2{}^* = r_1 r_2 e^{i(\alpha-\theta)}$ ***AG***

Note: Accept working in modulus-argument form

[2 marks]

(b) $\text{Re}\left(z_1 z_2{}^*\right) = r_1 r_2 \cos(\alpha-\theta) \quad (=0)$ ***A1***

$\alpha - \theta = \arccos 0 \ \left(r_1, r_2 > 0\right)$

$\alpha - \theta = \frac{\pi}{2}$ (as $0 < \alpha - \theta < \pi$) ***A1***

so Z_1OZ_2 is a right-angled triangle ***AG***

[2 marks]

(c) (i) **EITHER**

$\frac{z_1}{z_2}\left(=\frac{r_1}{r_2}e^{i(\alpha-\theta)}\right) = e^{i\frac{\pi}{3}}$ (since $r_1 = r_2$) ***(M1)***

OR

$z_1 = r_2 e^{i\left(\theta+\frac{\pi}{3}\right)} \ \left(= r_2 e^{i\theta} e^{i\frac{\pi}{3}}\right)$ ***(M1)***

THEN

$z_1 = z_2 e^{i\frac{\pi}{3}}$ ***A1***

Note: Accept working in either modulus-argument form to obtain $z_1 = z_2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or in Cartesian form to obtain $z_1 = z_2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$.

continued…

Question 12 continued

(ii) substitutes $z_1 = z_2 e^{i\frac{\pi}{3}}$ into $z_1{}^2 + z_2{}^2$ ***M1***

$z_1{}^2 + z_2{}^2 = z_2{}^2 e^{i\frac{2\pi}{3}} + z_2{}^2 \quad \left(= z_2{}^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$ ***A1***

EITHER

$e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$ ***A1***

OR

$z_2{}^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) = z_2{}^2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i + 1 \right)$

$= z_2{}^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$ ***A1***

THEN

$z_1{}^2 + z_2{}^2 = z_2{}^2 e^{i\frac{\pi}{3}}$

$= z_2 \left(z_2 e^{i\frac{\pi}{3}} \right)$ and $z_2 e^{i\frac{\pi}{3}} = z_1$ ***A1***

so $z_1{}^2 + z_2{}^2 = z_1 z_2$ ***AG***

Note: For candidates who work on the LHS and RHS separately to show equality, award ***M1A1*** for $z_1{}^2 + z_2{}^2 = z_2{}^2 e^{i\frac{2\pi}{3}} + z_2{}^2 \quad \left(= z_2{}^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$, ***A1*** for $z_1 z_2 = z_2{}^2 e^{i\frac{\pi}{3}}$ and ***A1*** for $e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$. Accept working in either modulus-argument form or in Cartesian form.

[6 marks]

continued…

Question 12 continued

(d) **METHOD 1**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$a^2 = z_1{}^2 + z_2{}^2 + 2z_1 z_2$ ***A1***

$a^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

METHOD 2

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$\left(z_1 + z_2\right)^2 = z_1{}^2 + z_2{}^2 + 2z_1 z_2$ ***A1***

$\left(z_1 + z_2\right)^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ and $z_1 + z_2 = -a$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

[5 marks]

(e) $a^2 - 3 \times 12 = 0$

$a = \pm 6 \ \left(\Rightarrow z^2 \pm 6z + 12 = 0\right)$ ***A1***

for $a = -6$:

$z_1 = 3 + \sqrt{3}\mathrm{i},\ z_2 = 3 - \sqrt{3}\mathrm{i}$ and $\alpha - \theta = -\frac{5\pi}{3}$ which does not satisfy $0 < \alpha - \theta < \pi$ ***R1***

for $a = 6$:

$z_1 = -3 - \sqrt{3}\mathrm{i},\ z_2 = -3 + \sqrt{3}\mathrm{i}$ and $\alpha - \theta = \frac{\pi}{3}$ ***A1***

so (for $0 < \alpha - \theta < \pi$), only one equilateral triangle can be formed from point O and the two roots of this equation ***AG***

[3 marks]

Total [18 marks]

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 1

36 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. $\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int \left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

[5 marks]

2. (a) $\text{IQR} = 10 - 6\ (= 4)$ ***(A1)***

attempt to find $Q_3 + 1.5 \times \text{IQR}$ ***(M1)***

$10 + 6$

16 ***A1***

[3 marks]

(b) (i) choosing $c = \frac{1}{2}a - 9$ ***(M1)***

$\frac{1}{2} \times 42 - 9$

$= 12$ (years old) ***A1***

(ii) attempt to solve system by substitution or elimination ***(M1)***

34 (years old) ***A1***

[4 marks]

Total [7 marks]

3. (a) $(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

(b) $\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognizing to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note:	Do not award the final ***A1*** if any additional solutions are seen. Award ***A1A0*** for correct answers in degrees. Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

Total [7 marks]

4. evidence of using product rule ***(M1)***

$\frac{dy}{dx} = (2x-1)\times\left(ke^{kx}\right) + 2\times e^{kx} \quad \left(= e^{kx}(2kx - k + 2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\frac{dy}{dx}$ at $x = 1 \Rightarrow ke^{k} + 2e^{k}$

OR

slope of tangent is $5e^{k}$

their $\frac{dy}{dx}$ at $x = 1$ equals the slope of $y = 5e^{k}x \left(= 5e^{k}\right)$ (seen anywhere) ***(M1)***

$ke^{k} + 2e^{k} = 5e^{k}$

$k = 3$ ***A1***

[5 marks]

5. (a) translation (shift) by $\frac{3\pi}{2}$ to the right OR positive horizontal direction by $\frac{3\pi}{2}$ ***A1***

translation (shift) by q upwards OR positive vertical direction by q ***A1***

> **Note**: accept translation by $\begin{pmatrix} \frac{3\pi}{2} \\ q \end{pmatrix}$
>
> Do not accept 'move' for translation/shift.

[2 marks]

(b) **METHOD 1**

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 (may be seen in sketch) ***(M1)***

$-4+2.5+q \geq 7$

$q \geq 8.5$ (accept $q=8.5$) ***A1***

substituting $x=0$ and their $q(=8.5)$ to find r ***(M1)***

$(r=)\quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+8.5$

$4+2.5+8.5$ ***(A1)***

smallest value of r is 15 ***A1***

continued…

Question 5 continued

METHOD 2

substituting $x=0$ to find an expression (for r) in terms of q ***(M1)***

$\left(g(0)=r=\right) \quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+q$

$\left(r=\right)\ 6.5+q$ ***A1***

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 ***(M1)***

$-4+2.5+q\geq 7$

$-4+2.5+(r-6.5)\geq 7$ (accept =) ***(A1)***

smallest value of r is 15 ***A1***

METHOD 3

$4\sin\left(x-\frac{3\pi}{2}\right)+2.5+q=4\cos x+2.5+q$ ***A1***

y-intercept of $4\cos x+2.5+q$ is a maximum ***(M1)***

amplitude of $g(x)$ is 4 ***(A1)***

attempt to find least maximum ***(M1)***

$r=2\times 4+7$

smallest value of r is 15 ***A1***

[5 marks]

Total [7 marks]

6. EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = {}^{n}C_{r}\left(8x^{3}\right)^{n-r}\left(-\frac{1}{2x}\right)^{r}$ OR $T_{r+1} = {}^{n}C_{n-r}\left(8x^{3}\right)^{r}\left(-\frac{1}{2x}\right)^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n - r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $(r = 3, 6, 9, \ldots)$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, k \in \mathbb{Z}^{+}$ ***A1***

Note:	Accept n is a (positive) multiple of 4 or $n = 4, 8, 12, \ldots$
	Do not accept $n = 4, 8, 12$
Note:	Award full marks for a correct answer using trial and error approach showing $n = 4, 8, 12, \ldots$ and for recognizing that this pattern continues.

[5 marks]

7. (a) attempt to integrate $\frac{k}{\sqrt{4-3x^2}}$ ***(M1)***

$=k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$ ***A1***

Note: Award ***(M1)A0*** for $\arcsin\left(\frac{\sqrt{3}}{2}x\right)$.

Condone absence of k up to this stage.

equating their integrand to 1 ***M1***

$k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]_0^1=1$

$k=\frac{3\sqrt{3}}{\pi}$ ***A1***

[4 marks]

continued…

Question 7 continued

(b) $\mathrm{E}(X)=\frac{3\sqrt{3}}{\pi}\int_0^1 \frac{x}{\sqrt{4-3x^2}}\,\mathrm{d}x$ ***A1***

Note: Condone absence of limits if seen at a later stage.

EITHER

attempt to integrate by inspection ***(M1)***

$=\frac{3\sqrt{3}}{\pi}\times-\frac{1}{6}\int -6x\left(4-3x^2\right)^{-\frac{1}{2}}\mathrm{d}x$

$=\frac{3\sqrt{3}}{\pi}\left[-\frac{1}{3}\sqrt{4-3x^2}\right]_0^1$ ***A1***

Note: Condone the use of k up to this stage.

OR

for example, $u=4-3x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-6x$

Note: Other substitutions may be used. For example, $u=-3x^2$.

$=-\frac{\sqrt{3}}{2\pi}\int_4^1 u^{-\frac{1}{2}}\,\mathrm{d}u$ ***M1***

Note: Condone absence of limits up to this stage.

$=-\frac{\sqrt{3}}{2\pi}\left[2\sqrt{u}\right]_4^1$ ***A1***

Note: Condone the use of k up to this stage.

THEN

$=\frac{\sqrt{3}}{\pi}$ ***A1***

Note: Award ***A0M1A1A0*** for their $k\left[-\frac{1}{3}\sqrt{4-3x^2}\right]$ or $k\left[-2\sqrt{u}\right]$ for working with incorrect or no limits.

[4 marks]

Total [8 marks]

8. Assume that a and b are both odd. ***M1***

> **Note**: Award ***M0*** for statements such as "let a and b be both odd".
>
> **Note**: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m+1$ and $b = 2n+1$ ***A1***

$a^2 + b^2 \equiv (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4\left(m^2 + m + n^2 + n\right) + 2$ ***(A1)***

($4\left(m^2 + m + n^2 + n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

> **Note**: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

9. (a) $z_1 z_2 = (1+b\text{i})\left(\left(1-b^2\right)-(2b)\text{i}\right)$

$= \left(1-b^2-2\text{i}^2b^2\right)+\text{i}\left(-2b+b-b^3\right)$ ***M1***

$= \left(1+b^2\right)+\text{i}\left(-b-b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1+b^2$ and ***A1*** for $-b\text{i}-b^3\text{i}$.

[3 marks]

(b) $\arg(z_1 z_2) = \arctan\left(\dfrac{-b-b^3}{1+b^2}\right) = \dfrac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \dfrac{\pi}{4}$ (since $1+b^2 \neq 0, \text{for } b \in \mathbb{R}$) ***A1***

OR

$-b-b^3 = 1+b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

Total [6 marks]

Section B

10. (a) (i) **EITHER**

attempt to use a ratio from consecutive terms ***M1***

$$\frac{p\ln x}{\ln x}=\frac{\frac{1}{3}\ln x}{p\ln x} \quad \text{OR} \quad \frac{1}{3}\ln x=(\ln x)r^2 \quad \text{OR} \quad p\ln x=\ln x\left(\frac{1}{3p}\right)$$

Note: Candidates may use $\ln x^1+\ln x^p+\ln x^{\frac{1}{3}}+\ldots$ and consider the powers of x in geometric sequence.

Award ***M1*** for $\frac{p}{1}=\frac{\frac{1}{3}}{p}$.

OR

$r=p$ and $r^2=\frac{1}{3}$ ***M1***

THEN

$p^2=\frac{1}{3}$ OR $r=\pm\frac{1}{\sqrt{3}}$ ***A1***

$p=\pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2=\frac{1}{3}$ or $p^2=\frac{1}{3}$ with no other working seen.

continued…

Question 10 continued

(ii) **EITHER**

since, $|p| = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $|p| = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

Note: Accept r instead of p. Award ***R0*** if both values of p not considered.

(iii) $\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} \ \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1 \ (\Rightarrow \ln x = 2)$ ***A1***

$x = e^2$ ***A1***

[6 marks]

continued…

Question 10 continued

(b) (i) **METHOD 1**

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p\ln x - \ln x = \frac{1}{3}\ln x - p\ln x$ OR $\frac{1}{3}\ln x = \ln x + 2\left(p\ln x - \ln x\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.
Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$.

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p\ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

continued…

Question 10 continued

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \quad \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p\ln x - \ln x = -\frac{1}{3}\ln x$ ***A1***

$p\ln x = \frac{2}{3}\ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

(ii) $d = -\frac{1}{3}\ln x$ ***A1***

continued…

Question 10 continued

(iii) **METHOD 1**

$$S_n = \frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right]$$

attempt to substitute into S_n and equate to $\ln\left(\frac{1}{x^3}\right)$ ***(M1)***

$$\frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right] = \ln\left(\frac{1}{x^3}\right)$$

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2\ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n\ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right)\ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ (or equivalent)

> **Note:** Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to
> $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

continued…

Question 10 continued

METHOD 2

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} terms $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[12 marks]

Total [18 marks]

11. (a) **METHOD 1**

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x+z=-3$ and ***A1***

$-3x+z=44$ ***A1***

OR

$-5x+y=-7$ and ***A1***

$-5x+y=40$ ***A1***

OR

$3x-z=3$ and ***A1***

$3x-z=-\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3\neq 44$ or $-7\neq 40$ or $3\neq -\frac{79}{5}$) ***R1***

Note:	There are other possible pairs of equations in two variables. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in Π_3 ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32,$ a contradiction ***R1***

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

METHOD 3

attempt to eliminate a variable ***M1***

$-3y+5z=6$ ***A1***

$-3y+5z=100$ ***A1***

$0=94,$ a contradiction ***R1***

Note:	Accept other equivalent alternatives. Accept other valid methods. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

(b) (i) $\Pi_1 : 2 + 2 + 0 = 4$ and $\Pi_2 : 1 + 4 + 0 = 5$ ***A1***

(ii) **METHOD 1**

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note:	Award ***A1A0*** if "$r =$" is missing. Accept any multiple of the direction vector. Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$" only once.

continued…

Question 11 continued

METHOD 2

attempt to eliminate a variable from Π_1 and Π_2 ***M1***

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form) ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes Π_1 and Π_2.

[5 marks]

continued…

Question 11 continued

(c) **METHOD 1**

the line connecting L and Π_3 is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$$\boldsymbol{s} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$ ***A1***

substitute $(1-9t, -2+3t, -2t)$ in Π_3 ***M1***

$-9(1-9t)+3(-2+3t)-2(-2t)=32$

$94t = 47 \Rightarrow t = \dfrac{1}{2}$ ***A1***

attempt to find distance between $(1,-2,0)$ and their point $\left(-\dfrac{7}{2}, -\dfrac{1}{2}, -1\right)$ ***(M1)***

$$= \left|\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$$

$$= \frac{\sqrt{94}}{2}$$ ***A1***

[6 marks]

continued…

Question 11 continued

METHOD 2

unit normal vector equation of Π_3 is given by $\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$=\dfrac{32}{\sqrt{94}}$ ***A1***

let Π_4 be the plane parallel to Π_3 and passing through P,

then the normal vector equation of Π_4 is given by

$\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=-15$ ***M1***

unit normal vector equation of Π_4 is given by

$\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\dfrac{-15}{\sqrt{94}}$ ***A1***

distance between the planes is $\dfrac{32}{\sqrt{94}}-\dfrac{-15}{\sqrt{94}}$ ***(M1)***

$=\dfrac{47}{\sqrt{94}}\left(=\dfrac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Total [15 marks]

12. (a) **METHOD 1**

recognition of both known series ***(M1)***

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \text{ and } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$$ ***(A1)***

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

METHOD 2

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \cos x + e^x \sin x$$ ***A1***

$$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \left(= 2e^x \cos x\right)$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x$$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x\left(\cos x - \sin x\right)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \ldots$$

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

continued…

Question 12 continued

(b) $e^{x^2}\sin\left(x^2\right)=x^2+x^4+\frac{1}{3}x^6+\dots$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$$\int_0^1 e^{x^2}\sin(x^2)\,dx\approx\int_0^1\left(x^2+x^4+\frac{1}{3}x^6\right)dx$$

Note: Condone absence of limits up to this stage.

$$=\left[\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{21}\right]_0^1$$ ***A1***

$$=\frac{61}{105}$$ ***A1***

[4 marks]

continued…

Question 12 continued

(c) (i) attempt to use product rule at least once ***M1***

$g'(x) = e^x \cos x - e^x \sin x$ ***A1***

$g''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \left(= -2e^x \sin x\right)$ ***A1***

EITHER

$2\left(g'(x) - g(x)\right) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right) = -2e^x \sin x$ ***A1***

OR

$g''(x) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right)$ ***A1***

THEN

$g''(x) = 2\left(g'(x) - g(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-2e^x \sin x$.

(ii) $g'''(x) = 2\left(g''(x) - g'(x)\right)$ ***A1***

$g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-4e^x \cos x$.

[5 marks]

continued…

Question 12 continued

(d) attempt to substitute $x=0$ into a derivative **(M1)**

$g(0)=1,\ g'(0)=1,\ g''(0)=0$ **A1**

$g'''(0)=-2,\ g^{(4)}(0)=-4$ **(A1)**

attempt to substitute into Maclaurin formula **(M1)**

$g(x)=1+x-\frac{2}{3!}x^3-\frac{4}{4!}x^4+\ldots\left(=1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+\ldots\right)$ **A1**

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim_{x\to 0}\frac{e^x\cos x-1-x}{x^3}=\lim_{x\to 0}\frac{\left(1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+\ldots\right)-1-x}{x^3}$ **M1**

$=\lim_{x\to 0}\left(-\frac{1}{3}-\frac{1}{6}x+\ldots\right)$ **(A1)**

$=-\frac{1}{3}$ **A1**

Note: Condone the omission of $+\ldots$ in their working.

continued…

Question 12 continued

METHOD 2

$\lim_{x\to 0}\frac{e^x\cos x - 1 - x}{x^3} = \frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$= \lim_{x\to 0}\frac{e^x\cos x - e^x\sin x - 1}{3x^2}\left(= \lim_{x\to 0}\frac{g'(x)-1}{3x^2}\right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim_{x\to 0}\frac{-2e^x\sin x}{6x}\left(= \lim_{x\to 0}\frac{g''(x)}{6x}\right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim_{x\to 0}\frac{-2e^x\sin x - 2e^x\cos x}{6}\left(= \lim_{x\to 0}\frac{g'''(x)}{6}\right)$ ***A1***

$= -\frac{1}{3}$ ***A1***

[3 marks]

Total [21 marks]

Markscheme

November 2021

Mathematics: analysis and approaches

Higher level

Paper 1

38 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. METHOD 1

recognition that $y = \int \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)***

$y = \sin\left(x - \frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both x and y values into their integrated expression including c (***M1***)

$2 = \sin\frac{\pi}{2} + c$

$c = 1$

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

METHOD 2

$\int_{2}^{y} \mathrm{d}y = \int_{\frac{3\pi}{4}}^{x} \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)(A1)***

$y - 2 = \sin\left(x - \frac{\pi}{4}\right) - \sin\frac{\pi}{2}$ ***A1***

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

2. (a) (i) $x = 3$ ***A1***

(ii) $y = -2$ ***A1***

[2 marks]

(b) (i) $(-2, 0)$ (accept $x = -2$) ***A1***

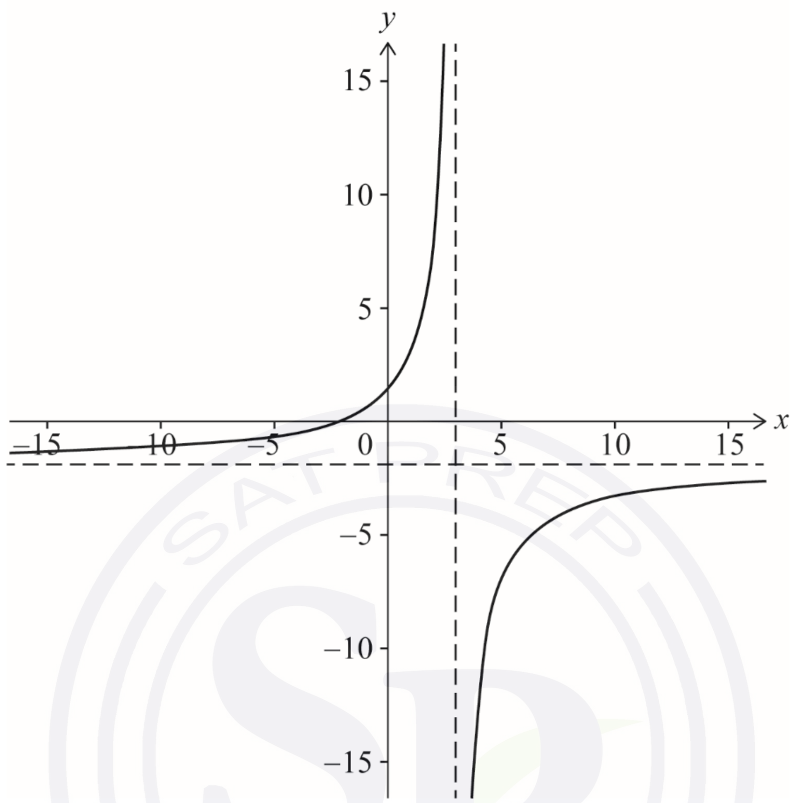
(ii) $\left(0, \frac{4}{3}\right)$(accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

[2 marks]

continued…

Question 2 continued.

(c)



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

continue…

Question 2 continued.

(d) **METHOD 1**

$\left(g(x)=\right)y=\dfrac{ax+4}{3-x}$

attempt to find x in terms of y ***(M1)***

OR exchange x and y and attempt to find y in terms of x

$3y-xy=ax+4$ ***A1***

$ax+xy=3y-4$

$x(a+y)=3y-4$

$x=\dfrac{3y-4}{y+a}$

$g^{-1}(x)=\dfrac{3x-4}{x+a}$ ***A1***

Note: Condone use of $y=$

$g(x)\equiv g^{-1}(x)$

$\dfrac{ax+4}{3-x}\equiv\dfrac{3x-4}{x+a}$

$\Rightarrow a=-3$ ***A1***

[4 marks]

continue…

Question 2 continued.

METHOD 2

$g(x)=\dfrac{ax+4}{3-x}$

attempt to find an expression for $g(g(x))$ and equate to x ***(M1)***

$gg(x)=\dfrac{a\left(\dfrac{ax+4}{3-x}\right)+4}{3-\left(\dfrac{ax+4}{3-x}\right)}=x$ ***A1***

$\dfrac{a(ax+4)+4(3-x)}{(9-3x)-(ax+4)}=x$

$\dfrac{a(ax+4)+4(3-x)}{5-(3+a)x}=x$

$a(ax+4)+4(3-x)=x(5-(3+a)x)$ ***A1***

equating coefficients of x^2 (or similar)

$a=-3$ ***A1***

[4 marks]

Total [9 marks]

3. attempt to use change the base ***(M1)***

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3\left(4x^3\right)$$

attempt to use the power rule ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3\left(4x^3\right)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ ***(M1)***

$$\log_3 \sqrt{x} = \log_3\left(4\sqrt{2}x^3\right)$$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen**.**

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$ ***(A1)***

$$x = \frac{1}{2}$$ ***A1***

[5 marks]

4. (a) valid approach to find $\mathrm{P}(R)$ ***(M1)***

tree diagram (must include probabilty of picking box) with correct required probabilities

OR $\mathrm{P}(R \cap B_1) + \mathrm{P}(R \cap B_2)$ OR $\mathrm{P}(R \mid B_1)\mathrm{P}(B_1) + \mathrm{P}(R \mid B_2)\mathrm{P}(B_2)$

$\frac{5}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}$ ***(A1)***

$\mathrm{P}(R) = \frac{9}{14}$ ***A1***

[3 marks]

(b) events A and R are not independent, since $\frac{9}{14} \cdot \frac{1}{2} \neq \frac{5}{14}$ OR $\frac{5}{7} \neq \frac{9}{14}$ OR $\frac{5}{9} \neq \frac{1}{2}$

OR an explanation e.g. different number of red balls in each box ***A2***

Note: Both conclusion and reasoning are required. Do not split the ***A2***.

[2 marks]

Total [5 marks]

5. (a) $f'(4)=6$ ***A1***

[1 mark]

(b) $f(4)=6\times 4-1=23$ ***A1***

[1 mark]

(c) $h(4)=f\left(g(4)\right)$ ***(M1)***

$h(4)=f(4^2-3\times 4)=f(4)$

$h(4)=23$ ***A1***

[2 marks]

(d) attempt to use chain rule to find h' ***(M1)***

$f'\left(g\left(x\right)\right)\times g'\left(x\right)$ **OR** $\left(x^2-3x\right)'\times f'\left(x^2-3x\right)$

$h'(4)=\left(2\times 4-3\right)f'\left(4^2-3\times 4\right)$ ***A1***

$=30$

$y-23=30\left(x-4\right)$ OR $y=30x-97$ ***A1***

[3 marks]

Total [7 marks]

6. (a) **METHOD 1**

attempt to write all LHS terms over a common denominator of $x-1$ ***(M1)***

$2x-3-\frac{6}{x-1}=\frac{2x(x-1)-3(x-1)-6}{x-1}$ OR $\frac{(2x-3)(x-1)}{x-1}-\frac{6}{x-1}$

$=\frac{2x^2-2x-3x+3-6}{x-1}$ OR $\frac{2x^2-5x+3}{x-1}-\frac{6}{x-1}$ ***A1***

$=\frac{2x^2-5x-3}{x-1}$ ***AG***

[2 marks]

METHOD 2

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x-3$ and remainder -6 ***A1***

$=2x-3-\frac{6}{x-1}$ as required. ***AG***

[2 marks]

continue…

Question 6 continued.

(b) consider the equation $\frac{2\sin^2 2\theta - 5\sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2\sin^2 2\theta - 5\sin 2\theta - 3 = 0$

EITHER

attempt to factorise in the form $(2\sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

Note: Accept any variable in place of $\sin 2\theta$.

$(2\sin 2\theta + 1)(\sin 2\theta - 3) = 0$

OR

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

THEN

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

Note: Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only.

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept 210 or 330) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

Note: Award ***A0*** if additional answers given.

[5 marks]

Total [7 marks]

7. (a) attempt to use discriminant $b^2 - 4ac(>0)$ ***M1***

$(2p)^2 - 4(3p)(1-p)(>0)$

$16p^2 - 12p(>0)$ ***(A1)***

$p(4p-3)(>0)$

attempt to find critical values $\left(p = 0,\ p = \frac{3}{4}\right)$ ***M1***

recognition that discriminant > 0 ***(M1)***

$p < 0$ or $p > \frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) $p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) ***M1***

$x = \frac{-8 \pm \sqrt{208}}{24}$

$x = \frac{-2 \pm \sqrt{13}}{6}$

$a = -2$ ***A1***

[2 marks]

Total [7 marks]

8. $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(\mathrm{e}^{\int\frac{2}{x}\mathrm{d}x}=\mathrm{e}^{2\ln x}\right)=x^2$ ***(A1)***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln 2x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)=\ln 2x$

$x^2y=\int \ln 2x\ \mathrm{d}x$

attempt to use integration by parts ***(M1)***

$x^2y=x\ln 2x-x\left(+c\right)$ ***A1***

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{c}{x^2}$

substituting $x=\frac{1}{2}, y=4$ into an integrated equation involving c ***M1***

$4=0-2+4c$

$\Rightarrow c=\frac{3}{2}$

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{3}{2x^2}$ ***A1***

[7 marks]

9. (a) attempt to expand binomial with negative fractional power ***(M1)***

$$\frac{1}{\sqrt{1+ax}} = (1+ax)^{-\frac{1}{2}} = 1 - \frac{ax}{2} + \frac{3a^2x^2}{8} + \ldots$$ ***A1***

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} + \ldots$$ ***A1***

$$\frac{1}{\sqrt{1+ax}} - \sqrt{1-x} = \frac{(1-a)}{2}x + \left(\frac{3a^2+1}{8}\right)x^2 + \ldots$$

attempt to equate coefficients of x or x^2 ***(M1)***

$$x : \quad \frac{1-a}{2} = 4b;\ x^2 : \frac{3a^2+1}{8} = b$$

attempt to solve simultaneously ***(M1)***

$$a = -\frac{1}{3}, b = \frac{1}{6}$$ ***A1***

[6 marks]

(b) $|x| < 1$ ***A1***

[1 mark]

Total [7 marks]

Section B

10. (a) (i) valid approach to find turning point ($v' = 0$, $-\frac{b}{2a}$, average of roots) ***(M1)***

$4 - 6t = 0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3}+2}{2}$

$t = \frac{2}{3}$ (s) ***A1***

(ii) attempt to integrate v ***(M1)***

$\int v \; \mathrm{d}t = \int \left(4 + 4t - 3t^2\right) \mathrm{d}t = 4t + 2t^2 - t^3 \left(+c\right)$ ***A1A1***

Note: Award ***A1*** for $4t + 2t^2$, ***A1*** for $-t^3$.

attempt to substitute their t into their solution for the integral ***(M1)***

distance $= 4\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$

$= \frac{8}{3} + \frac{8}{9} - \frac{8}{27}$ (or equivalent) ***A1***

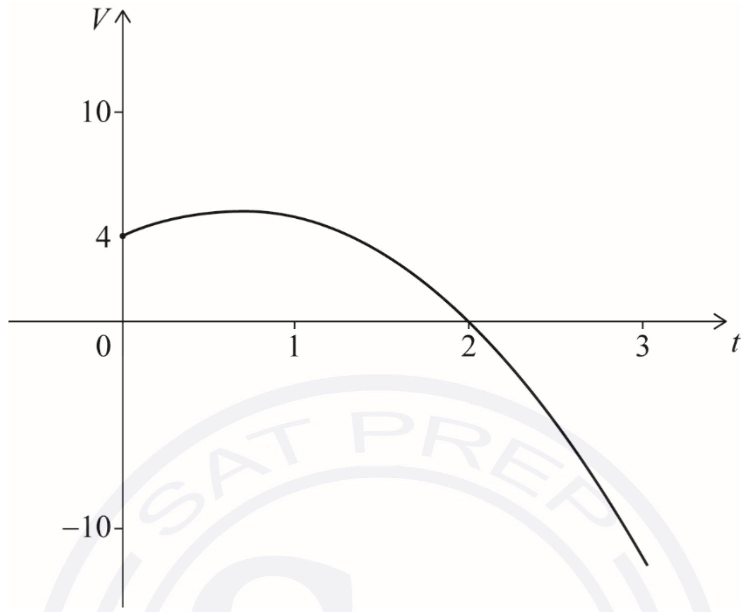
$= \frac{88}{27}$ (m) ***AG***

[7 marks]

continue…

Question 10 continued.

(b)



valid approach to solve $4+4t-3t^2=0$ (may be seen in part (a)) ***(M1)***

$(2-t)(2+3t)$ OR $\frac{-4\pm\sqrt{16+48}}{-6}$

correct x- intercept on the graph at $t=2$ ***A1***

Note: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0,4)$ ***A1***

Note: The 3 must be clearly indicated.

vertex in approximately correct place for $t=\frac{2}{3}$ and $v>4$ ***A1***

[4 marks]

continue…

Question 10 continued.

(c) recognising to integrate between 0 and 2, or 2 and 3 OR $\int_0^3 \left|4+4t-3t^2\right| \, \mathrm{d}t$ ***(M1)***

$\int_0^2 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=8$ ***A1***

$\int_2^3 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \, \mathrm{d}t - \int_2^3 v \, \mathrm{d}t$ OR $\int_0^2 v \, \mathrm{d}t + \left|\int_2^3 v \, \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

[5 marks]

Total [16 marks]

11. (a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x\right)$

$=(2x+2k)\mathrm{e}^x+\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2(k+1)x+k(k+1)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award **R1** only if three of the previous four marks have been awarded

[7 marks]

continue…

Question 11 continued.

(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n(n-1)\right]\mathrm{e}^x$ ***(M1)***

Note: For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{\mid x=0}=n(n-1)$ may be seen.

$f(0)=0, f'(0)=0, f''(0)=2, f'''(0)=6, f^{(4)}(0)=12$

use of $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$ ***(M1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

METHOD 2

'$x^2\times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

continue…

Question 11 continued.

(c) **METHOD 1**

attempt to substitute $x^2 e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2 e^x - x^2\right)^3}{x^9}$ ***M1***

$$\frac{\left(x^2 e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4 (+\ldots) - x^2\right)^3}{x^9}$$ ***(A1)***

EITHER

$$= \frac{\left(x^3 + \frac{1}{2}x^4 + \ldots\right)^3}{x^9}$$ ***A1***

$$= \frac{x^9 (+\text{higher order terms})}{x^9}$$

OR

$$\left(\frac{x^3 + \frac{1}{2}x^4 (+\ldots)}{x^3}\right)^3$$ ***A1***

$$\left(1 + \frac{1}{2}x(+\ldots)\right)^3$$

THEN

$= 1$ ($+$ higher order terms)

So $\lim_{x \to 0}\left[\frac{\left(x^2 e^x - x^2\right)^3}{x^9}\right] = 1$ ***A1***

[4 marks]

continue…

Question 11 continued.

METHOD 2

$$\lim_{x\to 0}\left[\frac{\left(x^2\mathrm{e}^x - x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2\mathrm{e}^x - x^2}{x^3}\right)^3$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{\mathrm{e}^x - 1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$= \lim_{x\to 0}\left(\frac{\mathrm{e}^x - 0}{1}\right)^3$$

$$= \left[\lim_{x\to 0}\mathrm{e}^x\right]^3$$

$$= 1$$ ***A1***

[4 marks]

Total [14 marks]

12. (a) (i) $\left(1+e^{i\frac{\pi}{6}}-1\right)^3$

$=\left(e^{i\frac{\pi}{6}}\right)^3$ ***A1***

$=e^{i\frac{\pi}{2}}$ ***A1***

$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$=i$ ***AG***

Note: Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii).

(ii) $(z-1)^3=e^{i\left(\frac{\pi}{2}+2\pi k\right)}$ ***(M1)***

$z-1=e^{i\left(\frac{\pi}{6}+\frac{4\pi k}{6}\right)}$ ***(M1)***

$(k=1)\Rightarrow \omega_2=1+e^{i\frac{5\pi}{6}}$ ***A1***

$(k=2)\Rightarrow \omega_3=1+e^{i\frac{9\pi}{6}}$ ***A1***

[6 marks]

continue…

Question 12 continued.

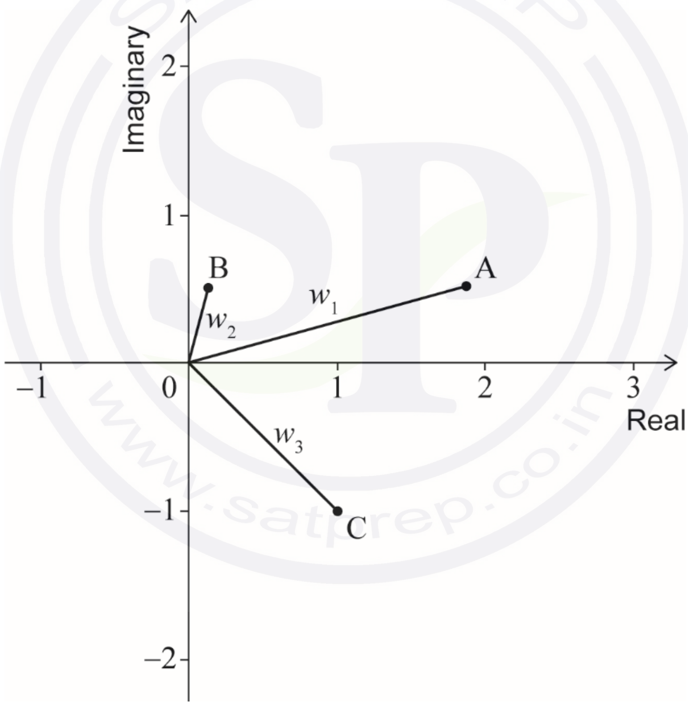
(b) **EITHER**

attempt to express $e^{i\frac{\pi}{6}}$, $e^{i\frac{5\pi}{6}}$, $e^{i\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

OR

attempt to express w_1, w_2 and w_3 in Cartesian form ***(M1)***

THEN



Note: To award ***A*** marks, it is not necessary to see A,B or C, the w_i , or the solid lines

A1A1A1

[4 marks]

continue…

Question 12 continued.

(c) valid attempt to find $\omega_1 - \omega_3$ (or $\omega_3 - \omega_1$) ***M1***

$$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - (1 - \mathrm{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \text{ OR } \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{2}$$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right|$ ***M1***

$$= \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$\mathrm{AC} = \sqrt{3}$ ***A1***

[3 marks]

continue…

Question 12 continued.

(d) **METHOD 1**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha-1}{\alpha} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$\alpha - 1 = \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$

$\alpha - \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 1$

$\alpha\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) = 1$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

continue…

Question 12 continued.

METHOD 2

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

continue…

Question 12 continued.

METHOD 3

LHS= $(z-1)^3=\left(\frac{1}{1-e^{i\frac{\pi}{6}}}-1\right)^3$

$$=\left(\frac{e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}}\right)^3$$

$$=\frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}\left(=\frac{i}{\frac{5}{2}-\frac{3\sqrt{3}}{2}+i\left(\frac{3\sqrt{3}}{2}-\frac{5}{2}\right)}\right)$$ ***M1A1***

Note: Award ***M1*** for applying de Moivre's theorem (may be seen in modulus- argument form.)

RHS= $iz^3=i\left(\frac{1}{1-e^{i\frac{\pi}{6}}}\right)^3$

$$=\frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}$$ ***A1***

$(z-1)^3=iz^3$ ***AG***

continue…

Question 12 continued.

METHOD 4

$(z-1)^3 = \mathrm{i}z^3$

$z^3 - 3z^2 + 3z - 1 = \mathrm{i}z^3$

$(1-\mathrm{i})z^3 - 3z^2 + 3z - 1 = 0$ ***(M1)***

$(1-\mathrm{i})\left(\dfrac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3 - 3\left(\dfrac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^2 + 3\left(\dfrac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right) - 1$

$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2 - \left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$ ***(A1)***

$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right) - \left(1-3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)$ ***A1***

$= 0$ ***AG***

Note: If the candidate does not interpret their conclusion, award ***(M1)(A1)A0*** as appropriate.

[3 marks]

continue…

Question 12 continued.

(e) **METHOD 1**

$$\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)}$$ ***M1***

$$=\frac{2}{2-\sqrt{3}-\mathrm{i}}$$ ***A1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{4-2\sqrt{3}+2\mathrm{i}}{\left(2-\sqrt{3}\right)^2+1}$$ ***A1***

$$=\frac{4-2\sqrt{3}+2\mathrm{i}}{8-4\sqrt{3}}$$ ***A1***

$$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}\mathrm{i}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

continue…

Question 12 continued.

METHOD 2

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{1}{\left(1-\cos\frac{\pi}{6}\right)-i\sin\frac{\pi}{6}}\times\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

$$=\frac{1}{2}+\frac{i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

continue…

Question 12 continued.

METHOD 3

attempt to multiply through by $-\frac{e^{-i\frac{\pi}{12}}}{e^{-i\frac{\pi}{12}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=-\frac{e^{-i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}-e^{-i\frac{\pi}{12}}}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=-\frac{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}$$ ***A1***

$$=-\frac{\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}}{2i\sin\frac{\pi}{12}}$$ ***A1***

$$=\frac{1}{2}-\frac{1}{2i}\cot\frac{\pi}{12}\ \left(=\frac{1}{2}+\frac{1}{2}i\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

[6 marks]

continue…

Question 12 continued.

METHOD 4

attempt to multiply through by $\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}-e^{i\frac{\pi}{6}}+1}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=\frac{1-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

attempt to re-write in Cartesian form ***M1***

$$=\frac{1-\frac{\sqrt{3}}{2}-\frac{1}{2}i}{2-\sqrt{3}}\left(=\frac{\frac{2-\sqrt{3}}{2}}{2-\sqrt{3}}+i\frac{\frac{1}{2}}{2-\sqrt{3}}\right)$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final ***A*** mark to be awarded

[6 marks]

Total [22 marks]

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. attempt to subtract squares of integers *(M1)*

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 \left(= 2n+1\right)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)(= 2n+1)$

THEN

$= n + n + 1$ =RHS ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

2. (a) attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4\times 2\times 2}}{4}\left(= \frac{5\pm 3}{4}\right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

Total [7 marks]

3. **EITHER**

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r = 2$ (or $r = 5$) ***(A1)***

THEN

${}^7C_2 = 21$ (or ${}^7C_5 = 21$) (seen anywhere) ***(A1)***

$21x^5k^2 = 63x^5$ ($21k^2 = 63$, $k^2 = 3$) ***A1***

$k = \pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k = \sqrt{3}$.

[5 marks]

4. (a) $\ln\left(x^2-16\right)=0$ ***(M1)***

$e^0=x^2-16(=1)$

$x^2=17$ OR $x=\pm\sqrt{17}$ ***(A1)***

$a=\sqrt{17}$ ***A1***

[3 marks]

(b) attempt to differentiate (must include $2x$ and/or $\frac{1}{x^2-16}$) ***(M1)***

$f'(x)=\frac{2x}{x^2-16}$ ***A1***

setting their derivative $=\frac{1}{3}$ ***M1***

$\frac{2x}{x^2-16}=\frac{1}{3}$

$x^2-16=6x$ ***OR*** $x^2-6x-16=0$ (or equivalent) ***A1***

valid attempt to solve their quadratic ***(M1)***

$x=8$ ***A1***

Note: Award ***A0*** if the candidate's final answer includes additional solutions (such as $x=-2,\ 8$).

[6 marks]
Total [9 marks]

5. **METHOD 1**

use of $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ on the LHS ***(M1)***

$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ OR $= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (|\boldsymbol{a}||\boldsymbol{b}|\cos\theta)^2$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$ ***AG***

METHOD 2

use of $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ on the RHS ***(M1)***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ OR $= (|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2$ ***A1***

$= |\boldsymbol{a}\times\boldsymbol{b}|^2$ ***AG***

Note: If candidates attempt this question using cartesian vectors, e.g

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ **and** $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ **,**

award full marks if fully developed solutions are seen.
Otherwise award no marks.

[4 marks]

6. METHOD 1

attempt to use the cosine rule to find the value of x ***(M1)***

$100 = x^2 + 4x^2 - 2(x)(2x)\left(\frac{3}{4}\right)$ ***A1***

$2x^2 = 100$

$x^2 = 50$ OR $x = \sqrt{50}$ $\left(= 5\sqrt{2}\right)$ ***A1***

attempt to find $\sin \hat{C}$ (seen anywhere) ***(M1)***

$\sin^2 \hat{C} + \left(\frac{3}{4}\right)^2 = 1$ or $x^2 + 3^2 = 4^2$ OR right triangle with side 3 and hypotenuse 4

$\sin \hat{C} = \frac{\sqrt{7}}{4}$ ***(A1)***

> **Note:** The marks for finding $\sin \hat{C}$ may be awarded independently of the first three marks for finding x.

correct substitution into the area formula using their value of x (or x^2) and their value of $\sin \hat{C}$ ***(M1)***

$A = \frac{1}{2} \times 5\sqrt{2} \times 10\sqrt{2} \times \frac{\sqrt{7}}{4}$ or $A = \frac{1}{2} \times \sqrt{50} \times 2\sqrt{50} \times \frac{\sqrt{7}}{4}$

$A = \frac{25\sqrt{7}}{2}$ ***A1***

continued…

METHOD 2

attempt to find the height, h, of the triangle in terms of x ***(M1)***

$h^2+\left(\frac{3}{4}x\right)^2=x^2$ OR $h^2+\left(\frac{5}{4}x\right)^2=10^2$ OR $h=\frac{\sqrt{7}}{4}x$ ***A1***

equating their expressions for either h^2 or h ***(M1)***

$x^2-\left(\frac{3}{4}x\right)^2=10^2-\left(\frac{5}{4}x\right)^2$ OR $\sqrt{100-\frac{25}{16}x^2}=\frac{\sqrt{7}}{4}x$ (or equivalent) ***A1***

$x^2=50$ OR $x=\sqrt{50}$ $\left(=5\sqrt{2}\right)$ ***A1***

correct substitution into the area formula using their value of x (or x^2) ***(M1)***

$A=\frac{1}{2}\times2\sqrt{50}\times\frac{\sqrt{7}}{4}\sqrt{50}$ OR $A=\frac{1}{2}\left(2\times5\sqrt{2}\right)\left(\frac{\sqrt{7}}{4}5\sqrt{2}\right)$

$A=\frac{25\sqrt{7}}{2}$ ***A1***

Total [7 marks]

7. $\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta\left(\alpha+\beta\right)=-3k$ (***A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\ \left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\ \left(=\sqrt{24}\right)\left(k>0\right)$ ***A1***

Note: Award ***A0*** for $k=\pm2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

[5 marks]

8. (a) **METHOD 1**

setting at least two components of l_1 and l_2 equal *M1*

$$3+2\lambda = 2+\mu \quad (1)$$
$$2-2\lambda = -\mu \quad (2)$$
$$-1+2\lambda = 4+\mu \quad (3)$$

attempt to solve two of the equations eg. adding (1) and (2) *M1*

gives a contradiction (no solution), eg $5=2$ *R1*

so l_1 and l_2 do not intersect *AG*

Note: For an error within the equations award ***M0M1R0.***   **Note:** The contradiction must be correct to award the ***R1.***

[3 marks]

METHOD 2

l_1 and l_2 are parallel, so l_1 and l_2 are either identical or distinct. *R1*

Attempt to subtract two position vectors from each line,

e.g. $\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\0\\4\end{pmatrix}\left(=\begin{pmatrix}1\\2\\-5\end{pmatrix}\right)$ *M1*

$\begin{pmatrix}3\\2\\-1\end{pmatrix}\neq k\begin{pmatrix}1\\-1\\1\end{pmatrix}$ *A1*

[3 marks]

(b) **METHOD 1**

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be $(3,2,-1)$ on l_1 and let B be $(2,0,4)$ on l_2

Attempt to find vector $\overrightarrow{\mathrm{AB}}\left(=\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right)$ *(M1)*

Distance required is $\dfrac{\left|\boldsymbol{v}\times\overrightarrow{\mathrm{AB}}\right|}{|\boldsymbol{v}|}$ *M1*

continued…

Question 8 continued

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right|$ ***(A1)***

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}3\\6\\3\end{pmatrix}\right|$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be a fixed point on l_1 eg $(3,2,-1)$ and let B be a general point on l_2 $(2+\mu,-\mu,4+\mu)$

attempt to find vector $\overrightarrow{\mathrm{AB}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}\ (\mu\in\mathbb{R})$ ***A1***

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(-1+\mu)^2+(-2-\mu)^2+(5+\mu)^2}\ \left(=\sqrt{3\mu^2+12\mu+30}\right)$ ***M1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\left|\overrightarrow{\mathrm{AB}}\right|^2\right)=0 \Rightarrow 6\mu+12=0 \Rightarrow \mu=-2$ ***A1***

OR

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{3(\mu+2)^2+18}$ to obtain $\mu=-2$ ***A1***

THEN

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

continued…

Question 8 continued

METHOD 3

let A be $(3,2,-1)$ on l_1 and let B be $(2+\mu,-\mu,4+\mu)$ on l_2 ***(M1)***

(or let A be $(2,0,4)$ on l_2 and let B be $(3+2\lambda,2-2\lambda,-1+2\lambda)$ on l_1)

$\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $(\mu\in\mathbb{R})$ (or $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}$) ***A1***

$\begin{pmatrix}\mu-1\\-\mu-2\\\mu+5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ (or $\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$) ***M1***

$\mu=-2$ or $\lambda=1$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]
Total [8 marks]

9. $u = \sin x \Rightarrow \mathrm{d}u = \cos x \, \mathrm{d}x$ (or equivalent) ***A1***

$= \int \frac{u}{u^2 - u - 2} \mathrm{d}u$ ***A1***

attempt to use partial fractions ***M1***

$\left(\frac{u}{(u+1)(u-2)} \equiv \frac{A}{u+1} + \frac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1) \right)$

Valid attempt to solve for A and B ***(M1)***

$A = \frac{1}{3}$ and $B = \frac{2}{3}$ ***A1***

$\frac{u}{(u+1)(u-2)} \equiv \frac{1}{3(u+1)} + \frac{2}{3(u-2)}$

$\int \left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)} \right) \mathrm{d}u = \frac{1}{3} \ln|u+1| + \frac{2}{3} \ln|u-2| (+C)$ (or equivalent) ***A1***

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \frac{1}{3} \ln|\sin x + 1| + \frac{2}{3} \ln|\sin x - 2| + C$ ***A1***

Note: Condone further simplification of the correct answer.

[7 marks]

Section B

10. (a) $6+6\cos x=0$ (or setting their $f'(x)=0$) ***(M1)***

$\cos x=-1$ (or $\sin x=0$)

$x=\pi,\ x=3\pi$ ***A1A1***

[3 marks]

(b) attempt to integrate $\int_{\pi}^{3\pi}(6+6\cos x)\,dx$ ***(M1)***

$=[6x+6\sin x]_{\pi}^{3\pi}$ ***A1A1***

substitute their limits into their integrated expression and subtract ***(M1)***

$=(18\pi+6\sin 3\pi)-(6\pi+6\sin\pi)$

$=(6(3\pi)+0)-(6\pi+0)$ $(=18\pi-6\pi)$ ***A1***

area $=12\pi$ ***AG***

[5 marks]

(c) attempt to substitute into the formula for surface area (including base) ***(M1)***

$\pi(2^2)+\pi(2)(l)=12\pi$ ***(A1)***

$4\pi+2\pi l=12\pi$

$2\pi l=8\pi$

$l=4$ ***A1***

[3 marks]

continued…

Question 10 continued

(d) valid attempt to find the height of the cone ***(M1)***
e.g. $2^2 + h^2 = (\text{their } l)^2$

$h = \sqrt{12}$ $\left(= 2\sqrt{3}\right)$ ***(A1)***

attempt to use $V = \frac{1}{3}\pi r^2 h$ with their values substituted ***M1***

$\left(\frac{1}{3}\pi\left(2^2\right)\left(\sqrt{12}\right)\right)$

$\text{volume} = \frac{4\pi\sqrt{12}}{3}\left(= \frac{8\pi\sqrt{3}}{3} = \frac{8\pi}{\sqrt{3}}\right)$ ***A1***

[4 marks]
Total [15 marks]

11. (a) $\frac{dv}{dt} = -(1+v)$ ***(A1)***

$\int 1 dt = \int -\frac{1}{1+v} dv$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1+v_0)$

$t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t(1+v) = 1+v_0$

$1+v = (1+v_0)e^{-t}$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

continued…

OR

$$t - C = -\ln(1+v) \Rightarrow e^{t-C} = \frac{1}{(1+v)}$$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$$e^{-C} = \frac{1}{(1+v_0)} \Rightarrow C = \ln(1+v_0)$$

$$t - \ln(1+v_0) = -\ln(1+v) \Rightarrow t = \ln(1+v_0) - \ln(1+v)$$

$$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow e^t = \frac{1+v_0}{1+v}$$ ***A1***

$$e^t(1+v) = 1+v_0$$

$$1+v = (1+v_0)e^{-t}$$ ***A1***

$$v(t) = (1+v_0)e^{-t} - 1$$ ***AG***

OR

$$t - C = -\ln(1+v) \Rightarrow e^{-t+C} = 1+v$$ ***A1***

$ke^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$k = 1+v_0$

$e^{-t}(1+v_0) = 1+v$ ***A1***

$$v(t) = (1+v_0)e^{-t} - 1$$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

continued…

Question 11 continued

(b) (i) recognition that when $t = T, v = 0$ ***M1***

$(1+v_0)e^{-T} - 1 = 0 \Rightarrow e^{-T} = \dfrac{1}{1+v_0}$ ***A1***

$e^{T} = 1+v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = e^{T} - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ dt \ \left(= \int \left((1+v_0)e^{-t} - 1\right)dt\right)$ ***(M1)***

$= -(1+v_0)e^{-t} - t(+D)$ ***A1***

($t = 0, s = 0$ so) $D = 1+v_0$ ***A1***

$s(t) = -(1+v_0)e^{-t} - t + 1 + v_0$

at s_{max}, $e^{T} = 1+v_0 \Rightarrow T = \ln(1+v_0)$

Substituting into $s(t)\left(= -(1+v_0)e^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{max} = -(1+v_0)\left(\dfrac{1}{1+v_0}\right) - \ln(1+v_0) + v_0 + 1$ ***A1***

$\left(s_{max} = v_0 - \ln(1+v_0)\right)$

[7 marks]

(c) **METHOD 1**

$v(T-k) = (1+v_0)e^{-T}e^{k} - 1$ ***(M1)***

$= (1+v_0)\left(\dfrac{1}{1+v_0}\right)e^{k} - 1$ ***A1***

$= e^{k} - 1$ ***AG***

[2 marks]

METHOD 2

$v(T-k) = (1+v_0)e^{-(T-k)} - 1$

$= e^{T}e^{-(T-k)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= e^{k} - 1$ ***AG***

[2 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$v(T+k) = (1+v_0)\mathrm{e}^{-T}\mathrm{e}^{-k} - 1$ ***(A1)***

$= \mathrm{e}^{-k} - 1$ ***A1***

[2 marks]

METHOD 2

$v(T+k) = (1+v_0)\mathrm{e}^{-(T+k)} - 1$ ***(A1)***

$= e^{T}\mathrm{e}^{-(T+k)} - 1$

$= e^{T-T-k} - 1$

$= \mathrm{e}^{-k} - 1$ ***A1***

[2 marks]

(e) **METHOD 1**

$v(T-k) + v(T+k) = \mathrm{e}^{k} + \mathrm{e}^{-k} - 2$ ***A1***

attempt to express as a square ***M1***

$= \left(\mathrm{e}^{\frac{k}{2}} - \mathrm{e}^{-\frac{k}{2}}\right)^2 \ (\geq 0)$ ***A1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k) + v(T+k) = \mathrm{e}^{k} + \mathrm{e}^{-k} - 2$ ***A1***

Attempt to solve $\frac{\mathrm{d}}{\mathrm{d}k}\left(\mathrm{e}^{k} + \mathrm{e}^{-k}\right) = 0 \ (\Rightarrow k = 0)$ ***M1***

minimum value of 2, (when $k = 0$), hence $\mathrm{e}^{k} + \mathrm{e}^{-k} \geq 2$ ***R1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]
Total [20 marks]

12. (a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

[3 marks]

continued…

Question 12 continued

METHOD 3

$$\tan\left(\arctan\left(2x+1\right)\right)=\tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\dfrac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

Question 12 continued

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{n}{n+1}\right)$ for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $\sum_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\mathrm{RHS}$ and so $\mathrm{P}(1)$ is true **R1**

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)$ $\left(k\in\mathbb{Z}^+\right)$ **M1**

Note: Award ***M0*** for statements such as "let $n=k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ **(*M1*)**

$$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***A1***

$$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ ***M1***

$$=\arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2(k+1)^3-k}\right)$$ ***A1***

Note: Award ***A1*** for correct numerator, with $(k+1)$ factored. Denominator does not need to be simplified

$$=\arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2k^3+6k^2+5k+2}\right)$$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$=\arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{(k+2)\left(2k^2+2k+1\right)}\right)=\arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

continued…

Question 12 continued

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so
$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^{+}$ ***R1***

Note:	Award the final ***R1*** mark provided at least four of the previous marks have been awarded.
Note:	To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]
Total [19 marks]

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 1

26 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

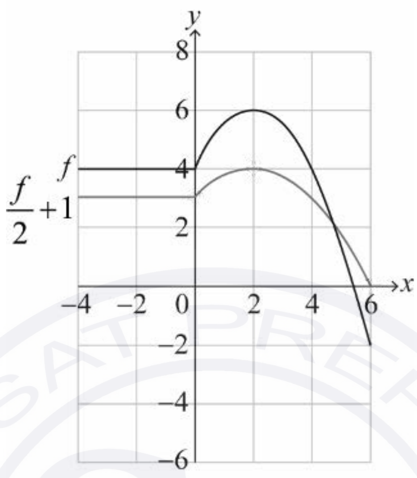
Section A

1. (a) (i) $f(2)=6$ ***A1***

(ii) $(f \circ f)(2)=-2$ ***A1***

[2 marks]

(b)



M1A1A1

Note: Award ***M1*** for an attempt to apply any vertical stretch or vertical translation,
A1 for a correct horizontal line segment between –4 and 0 (located roughly at $y=3$),
A1 for a correct concave down parabola including max point at (2,4) and for correct end points at (0,3) and (6,0) (within circles). Points do not need to be labelled.

[3 marks]
Total [5 marks]

2 **METHOD 1 (finding u_1 first, from S_8)**

$4(u_1+8)=8$ ***(A1)***

$u_1=-6$ ***A1***

$u_1+7d=8$ OR $4(2u_1+7d)=8$ (may be seen with their value of u_1) ***(A1)***

attempt to substitute their u_1 ***(M1)***

$d=2$ ***A1***

METHOD 2 (solving simultaneously)

$u_1+7d=8$ ***(A1)***

$4(u_1+8)=8$ OR $4(2u_1+7d)=8$ OR $u_1=-3d$ ***(A1)***

attempt to solve linear or simultaneous equations ***(M1)***

$u_1=-6, d=2$ ***A1A1***

[5 marks]

3. (a) attempt to use definition of outlier

$1.5 \times 20 + Q_3$ ***(M1)***

$1.5 \times 20 + U \geq 75$ $\left(\Rightarrow U \geq 45\right.$, accept $\left.U > 45\right)$ OR $1.5 \times 20 + Q_3 = 75$ ***A1***

minimum value of $U = 45$ ***A1***

[3 marks]

(b) attempt to use interquartile range ***(M1)***

$U - L = 20$ (may be seen in part (a)) OR $L \geq 25$ (accept $L > 25$)

minimum value of $L = 25$ ***A1***

[2 marks]

Total [5 marks]

4. (a) $f'(x) = -2(x-h)$ ***A1***

[1 mark]

(b) $g'(x) = e^{x-2}$ OR $g'(3) = e^{3-2}$ (may be seen anywhere) ***A1***

Note: The derivative of g must be explicitly seen, either in terms of x or 3.

recognizing $f'(3) = g'(3)$ ***(M1)***

$-2(3-h) = e^{3-2}$ $(= e)$

$-6+2h = e$ OR $3-h = -\frac{e}{2}$ ***A1***

Note: The final ***A1*** is dependent on one of the previous marks being awarded.

$h = \frac{e+6}{2}$ ***AG***

[3 marks]

(c) $f(3) = g(3)$ ***(M1)***

$-(3-h)^2 + 2k = e^{3-2} + k$

correct equation in k

EITHER

$-\left(3-\frac{e+6}{2}\right)^2 + 2k = e^{3-2} + k$ ***A1***

$k = e + \left(\frac{6-e-6}{2}\right)^2 \left(= e + \left(\frac{-e}{2}\right)^2\right)$ ***A1***

OR

$k = e + \left(3-\frac{e+6}{2}\right)^2$ ***A1***

$k = e + 9 - 3e - 18 + \frac{e^2 + 12e + 36}{4}$ ***A1***

THEN

$k = e + \frac{e^2}{4}$ ***AG***

[3 marks]
Total [7 marks]

5. (a)

> **Note:** Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS= $2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ =RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) attempt to factorise ***M1***

$(\cos x - \sin x)(2\sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \frac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\frac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\frac{\pi}{4}$ OR $\frac{\pi}{6}$ (accept $-\frac{\pi}{6}, \frac{7\pi}{6}$)

> **Note:** This ***(M1)(A1)*** is independent of the previous ***M1A1.***

$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{4}$ ***A2***

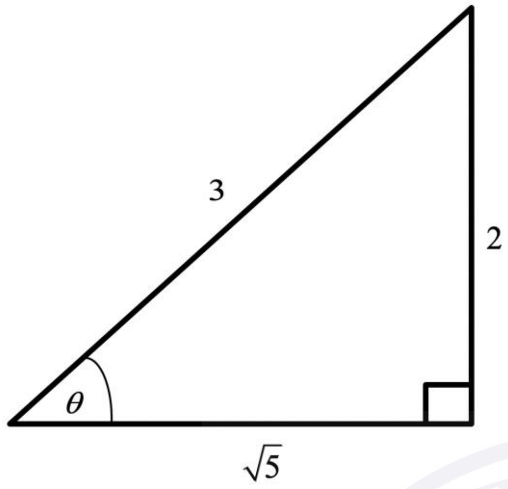
> **Note:** Award ***A1*** for any two correct (radian) answers.
> Award ***A1A0*** if additional values given with the four correct (radian) answers. Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

Total [8 marks]

6. **METHOD 1**

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

METHOD 2

Attempt to use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ ***M1***

$1 + \cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

7. **METHOD 1**

other two roots are $a-b\text{i}$ and $b-a\text{i}$ ***A1***

sum of roots $=-4$ and product of roots $=400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a+b\text{i}+a-b\text{i}+b+a\text{i}+b-a\text{i}=-4$

$2a+2b=-4\left(\Rightarrow a+b=-2\right)$ ***A1***

$(a+b\text{i})(a-b\text{i})(b+a\text{i})(b-a\text{i})=400$

$\left(a^2+b^2\right)^2=400$ ***A1***

$a^2+b^2=20$

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

METHOD 2

other two roots are $a-b\text{i}$ and $b-a\text{i}$ ***A1***

$\left(z-(a+b\text{i})\right)\left(z-(a-b\text{i})\right)\left(z-(b+a\text{i})\right)\left(z-(b-a\text{i})\right)(=0)$ ***A1***

$\left((z-a)^2+b^2\right)\left((z-b)^2+a^2\right)(=0)$

$\left(z^2-2az+a^2+b^2\right)\left(z^2-2bz+b^2+a^2\right)(=0)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a-2b=4$ and $\left(a^2+b^2\right)^2=400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

8. attempt to differentiate numerator and denominator ***M1***

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ ***(M1)***

$$=\frac{2}{3}$$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

9. (a) **METHOD 1**

B has one less pen to select *(M1)*

EITHER

A and B can be placed in 6×5 ways *(A1)*

C, D, E have 6 choices each *(A1)*

OR

A (or B), C, D, E have 6 choices each *(A1)*

B (or A) has only 5 choices *(A1)*

THEN

$5\times6^4\left(=6480\right)$ *A1*

METHOD 2

total number of ways $=6^5$ *(A1)*

number of ways with Amber and Brownie together $=6^4$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

6^5-6^4

$=5\times6^4\left(=6480\right)$ *A1*

[4 marks]

(b) **METHOD 1**

total number of ways $=6!\left(=720\right)$ *(A1)*

number of ways with Amber and Brownie sharing a boundary

$=2\times7\times4!\left(=336\right)$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

$720-336=384$ *A1*

METHOD 2

case 1: number of ways of placing A in corner pen

$3\times4\times3\times2\times1$

Four corners total no of ways is $4\times\left(3\times4\times3\times2\times1\right)=12\times4!\left(=288\right)$ *(A1)*

case 2: number of ways of placing A in the middle pen

$2\times4\times3\times2\times1$

two middle pens so $2\times\left(2\times4\times3\times2\times1\right)=4\times4!\left(=96\right)$ *(A1)*

attempt to add (may be seen in words) *(M1)*

total no of ways $=288+96$

$=16\times4!\left(=384\right)$ *A1*

[4 marks]

Total [8 marks]

Section B

10. (a) recognising probabilities sum to 1 *(M1)*

$p + p + p + \frac{1}{2}p = 1$

$p = \frac{2}{7}$ ***A1***

[2 marks]

(b) valid attempt to find $\mathrm{E}(X)$ *(M1)*

$1 \times p + 2 \times p + 3 \times p + 4 \times \frac{1}{2}p \left(= 8p\right)$

$\mathrm{E}(X) = \frac{16}{7}$ ***A1***

[2 marks]

(c) (i) $0 \le r \le 1$ ***A1***

(ii) Attempt to find a value of q *(M1)*

$0 \le 1 - 3q \le 1$ OR $r = 0 \Rightarrow q = \frac{1}{3}$ OR $r = 1 \Rightarrow q = 0$

$0 \le q \le \frac{1}{3}$ ***A1***

[3 marks]

(d) $\mathrm{E}(Y) = 1 \times q + 2 \times q + 3 \times q + 4 \times r \left(= 2 + 2r \text{ OR } 4 - 6q\right)$ *(A1)*

one correct boundary value ***A1***

$1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 4 \times 0 \left(= 2\right)$ OR

$1 \times 0 + 2 \times 0 + 3 \times 0 + 4 \times 1 \left(= 4\right)$ OR

$2 + 2(0) \left(= 2\right)$ OR

$2 + 2(1) \left(= 4\right)$ OR

$4 - 6(0) \left(= 4\right)$ OR $4 - 6\left(\frac{1}{3}\right) \left(= 2\right)$

$2 \le \mathrm{E}(Y) \le 4$ ***A1***

[3 marks]

continued…

Question 10 continued

(e) **METHOD 1**

evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

solving for either q or r ***M1***

$\frac{6}{7}(q+1-3q)=\frac{1}{2}$ OR $\frac{6}{7}\left(\frac{1-r}{3}+r\right)=\frac{1}{2}$ OR $3pq+3p(1-3q)=\frac{1}{2}$

OR $3p\left(\frac{1-r}{3}\right)+3pr=\frac{1}{2}$

EITHER two correct values

$q=\frac{5}{24}$ and $r=\frac{3}{8}$ ***A1A1***

OR one correct value

$q=\frac{5}{24}$ OR $r=\frac{3}{8}$ ***A1***

substituting their value for q or r ***A1***

$4-6\left(\frac{5}{24}\right)$ OR $2+2\left(\frac{3}{8}\right)$

THEN

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]

continued…

Question 10 continued

METHOD 2 (solving for $\mathrm{E}(Y)$)
evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

rearranging to make q the subject ***M1***

$q=\frac{4-\mathrm{E}(Y)}{6}$

$3pq+3p(1-3q)=\frac{1}{2}$ ***M1***

$\frac{6}{7}\times\left(\frac{4-\mathrm{E}(Y)}{6}\right)+\frac{6}{7}\left(1-3\left(\frac{4-\mathrm{E}(Y)}{6}\right)\right)=\frac{1}{2}$ ***A1***

$\frac{2(\mathrm{E}(Y)-1)}{7}=\frac{1}{2}$

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]
Total [16 marks]

11. (a) (i) $\frac{-1+1}{2}=0=3-3$ ***A1***

the point $(-1,0,3)$ lies on L_1. ***AG***

(ii) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2}=y=3-z=\lambda \Rightarrow x=2\lambda-1,\ y=\lambda,\ z=3-\lambda$ OR $\frac{x+1}{2}=\frac{y-0}{1}=\frac{z-3}{-1}$

correct direction vector $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r}=$ is omitted.

[4 marks]

(b) attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix}2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}a\\1\\-1\end{pmatrix}=(\pm)\sqrt{6}\sqrt{a^2+2}\cos 45^\circ$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a+2=\frac{(\pm)\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2}\left(\Rightarrow 2a+2=(\pm)\sqrt{3}\sqrt{a^2+2}\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$4a^2+8a+4=3a^2+6$

$a^2+8a-2=0$ ***A1***

attempt to solve their quadratic ***M1***

$a=\frac{-8\pm\sqrt{64+8}}{2}=\frac{-8\pm\sqrt{72}}{2}\left(=-4\pm 3\sqrt{2}\right)$ ***A1***

[8 marks]

continued…

Question 11 continued

(c) **METHOD 1**

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

Solutions exist unless $a - 2 = 0$

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right)\left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right)\left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

continued…

Question 11 continued

> **Note:** Award ***A1*** for any two correct coordinates seen or final answer in vector form.

[7 marks]
Total [19 marks]

12. (a) attempt to use the chain rule ***M1***

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ ***A1***

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ ***A1***

$= -\frac{1}{4\sqrt{(1+x)^3}}$ ***AG***

> **Note:** Award ***M1A0A0*** for $f'(x) = \frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n = 2$

$f''(x) = \left(-\frac{1}{4\sqrt{(1+x)^3}} = \right)\left(-\frac{1}{4}\right)^1 \frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ ***R1***

> **Note:** Award ***R0*** for not starting at $n = 2$. Award subsequent marks as appropriate.

assume true for $n = k$, (so $f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) ***M1***

> **Note:** Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

consider $n = k+1$

$\text{LHS} = f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}$ ***M1***

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2} - k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

EITHER

$\text{RHS} = f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^k \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$= \left(-\frac{1}{4}\right)^k \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!} = \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}$ $\left(= \frac{2(2k-1)(2k-3)!}{(k-2)!}\right)$

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

$$\left(= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}\right)$$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

continued…

Question 12 continued

$$=\left(-\frac{1}{4}\right)^{(k+1)-1}\frac{\left(2\left(k+1\right)-3\right)!}{(\left(k+1\right)-2)!}(1+x)^{\frac{1}{2}-(k+1)}=\text{RHS}$$

THEN

since true for $n=2$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n\in\mathbb{Z}, n\geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x)=\sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$$h'(x)=\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$$ ***A1***

$$h''(x)=m\left(\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right)+\frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx}-\frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$$ ***A1***

substituting $x=0$ into $h''(x)$ ***M1***

$$h''(0)=m^2+\frac{1}{2}m+\frac{1}{2}m-\frac{1}{4}\left(=m^2+m-\frac{1}{4}\right)$$ ***A1***

$$h(x)=h(0)+xh'(0)+\frac{x^2}{2!}h''(0)+\ldots$$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$$\frac{h''(0)}{2!}=\frac{7}{4}\quad\left(\Rightarrow h''(0)=\frac{7}{2}\right)$$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x)=(1+x)^{\frac{1}{2}}$ $\quad f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0)=-\frac{1}{4}$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$$f(x)=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\ldots$$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ ***A1***

THEN

$g(x)=1+mx+\frac{m^2}{2}x^2+\ldots$ ***(A1)***

$h(x)=\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right)\left(1+mx+\frac{m^2}{2}x^2+\ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}=\frac{7}{4}$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2 e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4} \quad \left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right) \quad \left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

Total [20 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 1

16 pages

International Baccalaureate®
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Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***A2***, *etc*., do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) or subpart(s). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then **FT** marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

Section A

1. attempt to substitute into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

Note: Accept use of Venn diagram or other valid method.

$0.6 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$ ***(A1)***

$\mathrm{P}(A \cap B) = 0.3$ (seen anywhere) ***A1***

attempt to substitute into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ ***(M1)***

$= \dfrac{0.3}{0.4}$

$\mathrm{P}(A \mid B) = 0.75 \left(= \dfrac{3}{4}\right)$ ***A1***

Total [5 marks]

2. (a) attempting to expand the LHS ***(M1)***

$\mathrm{LHS} = (4n^2 - 4n + 1) + (4n^2 + 4n + 1)$ ***A1***

$= 8n^2 + 2 (= \mathrm{RHS})$ ***AG***

[2 marks]

(b) ***METHOD 1***

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for all odd integers n) ***R1***

$8n^2 + 2 = 2(4n^2 + 1)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + (n+2)^2 = 2(n^2 + 2n + 2)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

Total [5 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{du}{dx} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\,dx = \int \frac{2}{\sqrt{u}}\,du$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

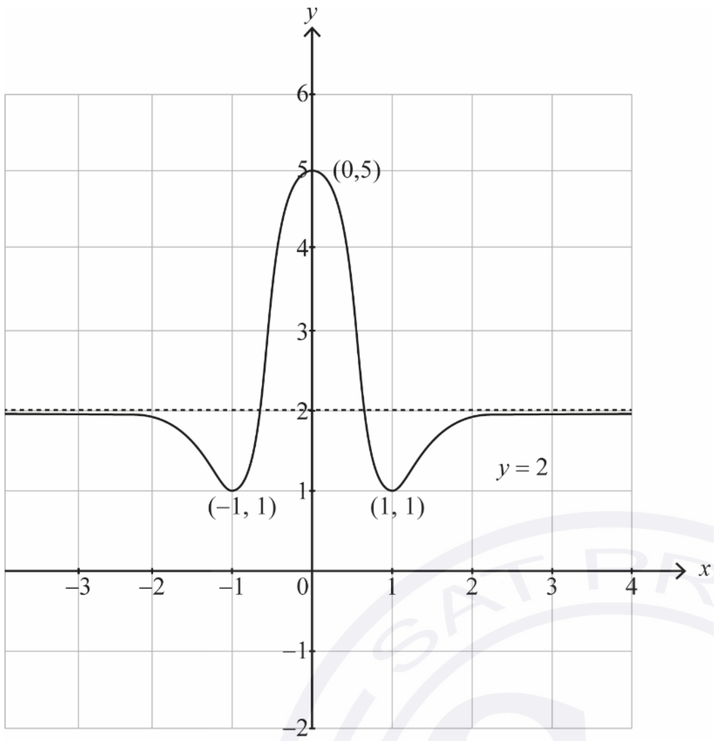
correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

4.



no y values below 1 **A1**

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ **A1**

$(\pm 1, 1)$ local minima **A1**

$(0, 5)$ local maximum **A1**

smooth curve and smooth stationary points **A1**

Total [5 marks]

5. (a) attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right) = 8\left(\frac{x+3}{4}\right) + 5$ ***A1***

$(g \circ f)(x) = 2x + 11$ ***AG***

[2 marks]

(b) attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a = 2 \times 4 + 11$ ***(A1)***

$a = 19$ ***A1***

[3 marks]

Total [5 marks]

6. (a) attempting to use the change of base rule ***M1***

$\log_9(\cos 2x + 2) = \dfrac{\log_3(\cos 2x + 2)}{\log_3 9}$ ***A1***

$= \dfrac{1}{2}\log_3(\cos 2x + 2)$ ***A1***

$= \log_3 \sqrt{\cos 2x + 2}$ ***AG***

[3 marks]

(b) $\log_3(2\sin x) = \log_3 \sqrt{\cos 2x + 2}$

$2\sin x = \sqrt{\cos 2x + 2}$ ***M1***

$4\sin^2 x = \cos 2x + 2$ (or equivalent) ***A1***

use of $\cos 2x = 1 - 2\sin^2 x$ ***(M1)***

$6\sin^2 x = 3$

$\sin x = (\pm)\dfrac{1}{\sqrt{2}}$ ***A1***

$x = \dfrac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x = \dfrac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

7. attempting integration by parts, *eg*

$u=\frac{\pi x}{36}, \mathrm{d}u=\frac{\pi}{36}\mathrm{d}x, \mathrm{d}v=\sin\left(\frac{\pi x}{6}\right)\mathrm{d}x, v=-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)$ ***(M1)***

$\mathrm{P}(0\le X\le 3)=\frac{\pi}{36}\left(\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3+\frac{6}{\pi}\int_0^3\cos\left(\frac{\pi x}{6}\right)\mathrm{d}x\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for a correct uv and ***A1*** for a correct $\int v\,\mathrm{d}u$.

attempting to substitute limits ***M1***

$\frac{\pi}{36}\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3=0$ ***(A1)***

so $\mathrm{P}(0\le X\le 3)=\frac{1}{\pi}\left[\sin\left(\frac{\pi x}{6}\right)\right]_0^3$ (or equivalent) ***A1***

$=\frac{1}{\pi}$ ***A1***

Total [7 marks]

8. recognition that the angle between the normal and the line is $60°$ (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60°=\frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9}\times\sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2}=\frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2}=4|p|$

attempt to square both sides ***M1***

$9(5+p^2)=16p^2\Rightarrow 7p^2=45$

$p=\pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

9. (a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) **Note:** Interchanging x and y can be done at any stage.

$y = \left(e^{x} - 3\right)^{2} - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \le \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \le x < 5$ ***A1***

[5 marks]

Total [8 marks]

Section B

10. (a) attempt to use quotient rule *(M1)*

correct substitution into quotient rule

$$f'(x) = \frac{5kx\left(\frac{1}{5x}\right) - k\ln 5x}{(kx)^2} \text{ (or equivalent)}$$ ***A1***

$$= \frac{k - k\ln 5x}{k^2x^2}, \left(k \in \mathbb{R}^+\right)$$ ***A1***

$$= \frac{1 - \ln 5x}{kx^2}$$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$$\frac{1 - \ln 5x}{kx^2} = 0$$

$\ln 5x = 1$ ***(A1)***

$x = \frac{e}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$$\frac{2\ln 5x - 3}{kx^3} = 0$$

$\ln 5x = \frac{3}{2}$ ***A1***

$5x = e^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \frac{1}{5}e^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx}\,\mathrm{d}x = \frac{1}{k}\int u\ \mathrm{d}u$ ***(A1)***

EITHER

$= \frac{u^2}{2k}$ ***A1***

so $\frac{1}{k}\int_1^{\frac{3}{2}} u\ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \frac{(\ln 5x)^2}{2k}$ ***A1***

so $\int_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}} \frac{\ln 5x}{kx}\,\mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}}$ ***A1***

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

11. (a) attempt to find modulus *(M1)*

$r = 2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant *(M1)*

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\text{i} = \sqrt{12}\text{e}^{\frac{5\pi\text{i}}{6}}\left(=2\sqrt{3}\text{e}^{\frac{5\pi\text{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\text{e}^{\frac{5\pi\text{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\text{e}^{-\frac{7\pi\text{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\text{e}^{\frac{17\pi\text{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

continued…

Question 11 continued

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles UOV, VOW and UOW **M1**

$$\text{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$$ **A1A1**

> **Note:** Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$$ (or equivalent) **A1**

[4 marks]

METHOD 2

$$UV^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$$ (or equivalent) **A1**

$$UV = \sqrt{3}\left(12^{\frac{1}{6}}\right)$$ (or equivalent) **A1**

attempting to find the area of UVW using $\text{Area} = \frac{1}{2}\times UV \times VW \times \sin\alpha$ for example **M1**

$$\text{Area} = \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$$

$$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$$ (or equivalent) **A1**

[4 marks]

(d) $u + v + w = 0$ **R1**

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + i\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + i\sin\frac{17\pi}{18}\right) = 0$$ **A1**

consideration of real parts **M1**

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{7\pi}{18}$ explicitly stated **A1**

$$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$$ **AG**

[4 marks]

Total [18 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x)=(\cos x)e^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x)=e^{\sin x}(\cos^2 x-\sin x)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0)=1;\ f'(0)=(\cos 0)e^{\sin 0}=1;\ f''(0)=e^{\sin 0}(\cos^2 0-\sin 0)=1$ ***A1***

substitution into the Maclaurin formula $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1+x+\frac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x)=(\cos x)e^{\sin x}(\cos^2 x-\sin x)-(\cos x)e^{\sin x}(2\sin x+1)$ (or equivalent) ***A2***

substituting $x=0$ into **their** $f'''(x)$ ***M1***

$f'''(0)=1(1-0)-1(0+1)=0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$e^{\sin x}=1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+...$

substituting Maclaurin series for $\sin x$ ***M1***

$$e^{\sin x}=1+\left(x-\frac{x^3}{3!}+...\right)+\frac{\left(x-\frac{x^3}{3!}+...\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+...\right)^3}{3!}+...$$ ***A1***

coefficient of x^3 is $-\frac{1}{3!}+\frac{1}{3!}=0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$$\mathrm{e}^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$$ ***A1***

substituting $\left(\mathrm{e}^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(\mathrm{e}^{3x} - 1\right) = \left(\mathrm{e}^{3x} - 1\right) - \frac{\left(\mathrm{e}^{3x} - 1\right)^3}{3} + \frac{\left(\mathrm{e}^{3x} - 1\right)^5}{5} - \ldots$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right)^3}{3} + \ldots$$ ***A1***

selecting correct terms from above ***M1***

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + \ldots}{3x + \frac{9x^2}{2} + \ldots}$$ ***A1***

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + \ldots}{3 + \frac{9x}{2} + \ldots}$$

$$= \frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x \to 0} \frac{(\cos x)\mathrm{e}^{\sin x}}{\frac{3\mathrm{e}^{3x}}{1 + \left(\mathrm{e}^{3x} - 1\right)^2}}$$ (or equivalent) ***A1***

$$= \frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]